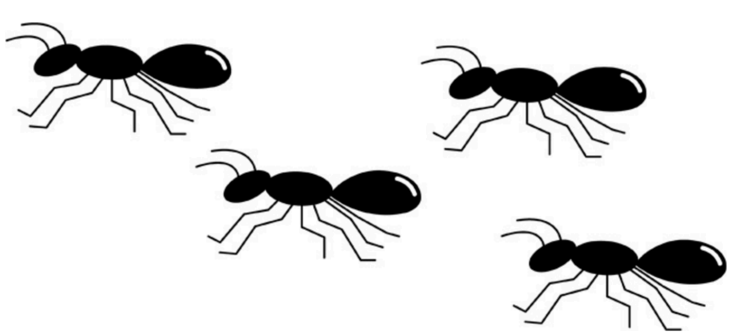
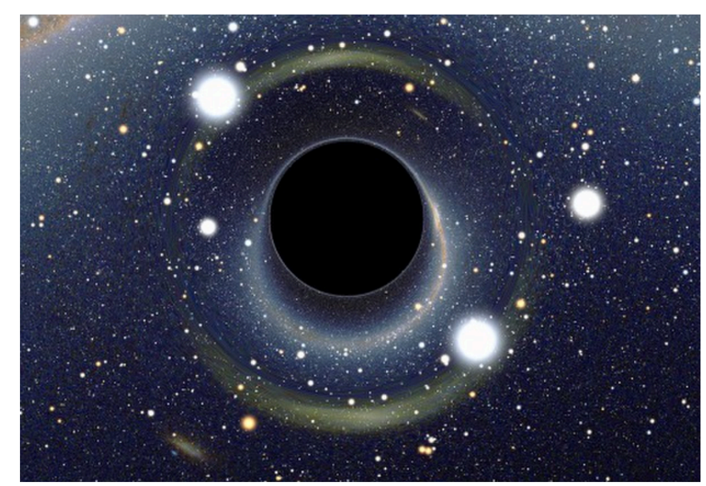
Evaporating Black Hole and Partial Deconfinement

Masanori Hanada
University of Southampton





28 Dec 2018 @ YITP, Kyoto

Holographic Principle

Black Hole

Quantum gravity

$\Longleftrightarrow$

‘Equivalent’

Non-gravitational systems

Matrix Model
Super Yang-Mills
SYK
....

Holographic Principle

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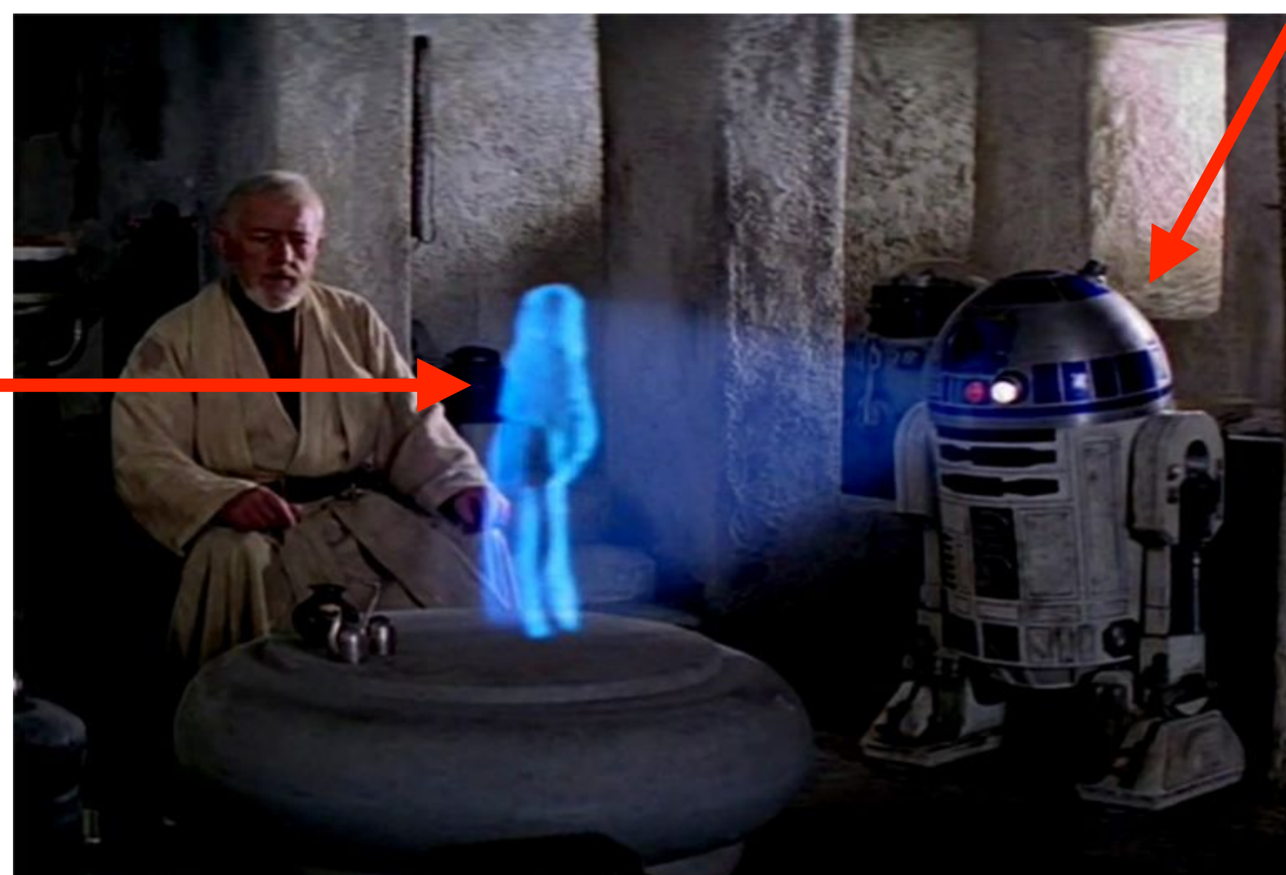
BH



- Our world with gravity is secretly non-gravitational.

arXiv.org > gr-qc > arXiv:gr-qc/9310026

General Relativity and Quantum Cosmology

Dimensional Reduction in Quantum Gravity

G. 't Hooft

arXiv.org > hep-th > arXiv:hep-th/9409089

High Energy Physics – Theory

The World as a Hologram

L. Susskind

arXiv.org > hep-th > arXiv:hep-th/9711200

Search or A

(Help | Advanc

High Energy Physics – Theory

The Large N Limit of Superconformal Field Theories and Supergravity

Juan M. Maldacena

We want to study it,
to learn about quantum gravity.

- Our world with gravity is secretly non-gravitational.

arXiv.org > gr-qc > arXiv:gr-qc/9310026

General Relativity and Quantum Cosmology

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Search or A
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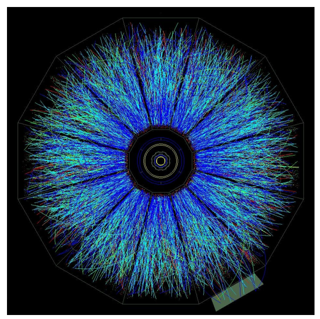
High Energy Physics – Theory

The Large N Limit of Superconformal Field Theories and Supergravity

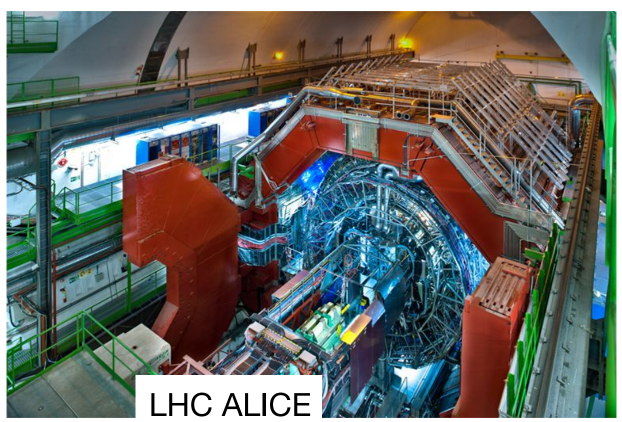
Juan M. Maldacena

Goal of Holography Program

Solve it.



Quark-gluon plasma

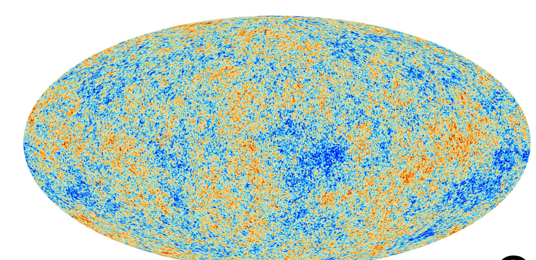


LHC ALICE

Gauge theory ～ QCD

And learn about it.

Black hole



CMB ?

Formation of quark-gluon plasma

Energy

“Finite-N, finite-coupling effects”

Formation of black hole

Mass of black hole

Corrections to Einstein gravity

QCD

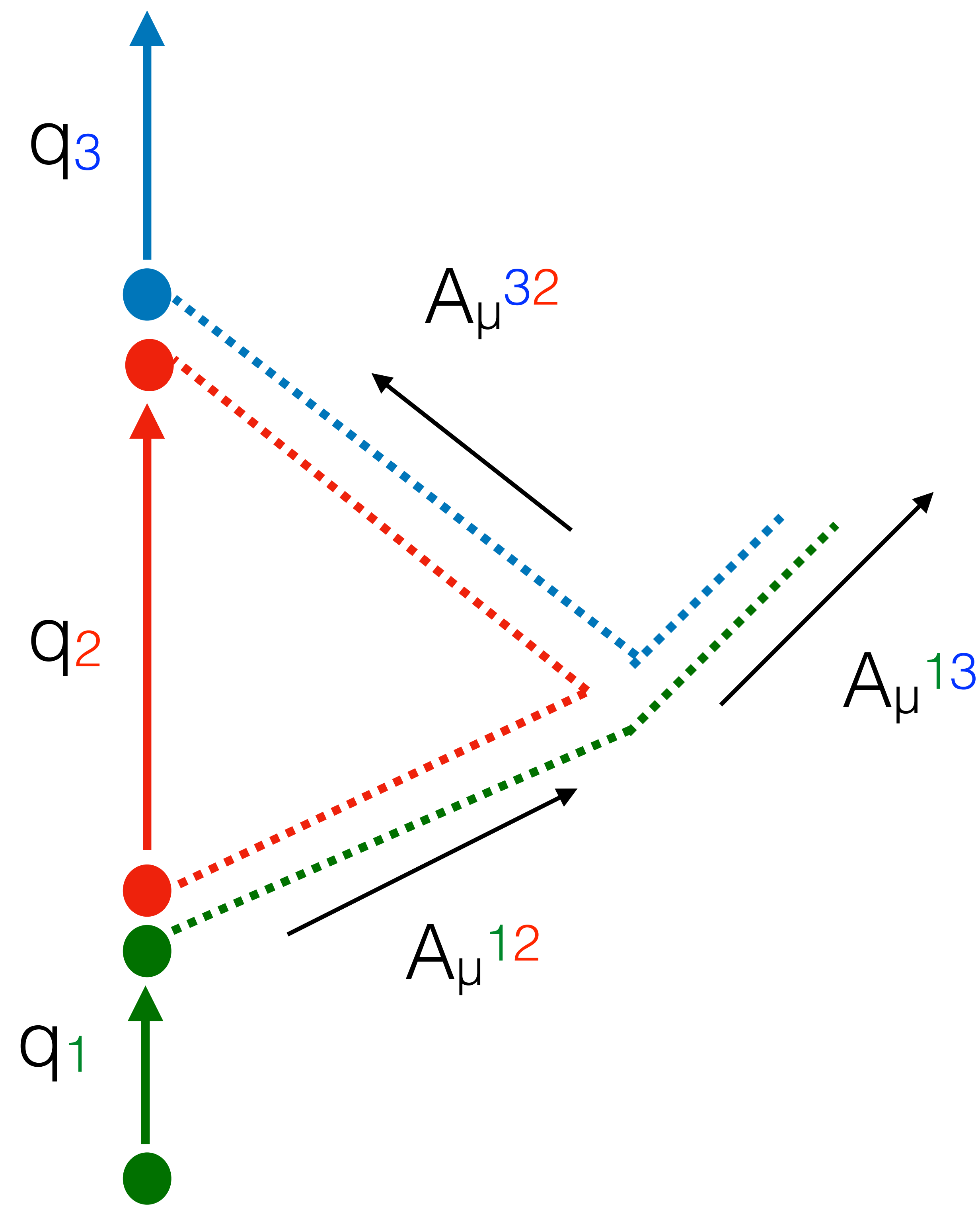
string force inside atoms

SU(3) gauge theory

3 colors

$$\begin{pmatrix} A_\mu^{11} & A_\mu^{12} & A_\mu^{13} \\ A_\mu^{21} & A_\mu^{22} & A_\mu^{23} \\ A_\mu^{31} & A_\mu^{32} & A_\mu^{33} \end{pmatrix} \qquad \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

gauge field (gluon) — quark



$\boxed{\text{QCD}}$

string force inside atoms

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$$\begin{pmatrix} A_\mu^{11} & A_\mu^{12} & A_\mu^{13} \\ A_\mu^{21} & A_\mu^{22} & A_\mu^{23} \\ A_\mu^{31} & A_\mu^{32} & A_\mu^{33} \end{pmatrix} \qquad \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$$

gauge field (gluon)

quark

$\boxed{\text{Supersymmetric Gauge Theory}}$

SU(N) gauge theory

N colors

$$\begin{pmatrix} A_\mu^{11} \ldots\ldots A_\mu^{1N} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ A_\mu^{N1} \ldots\ldots A_\mu^{NN} \end{pmatrix}$$

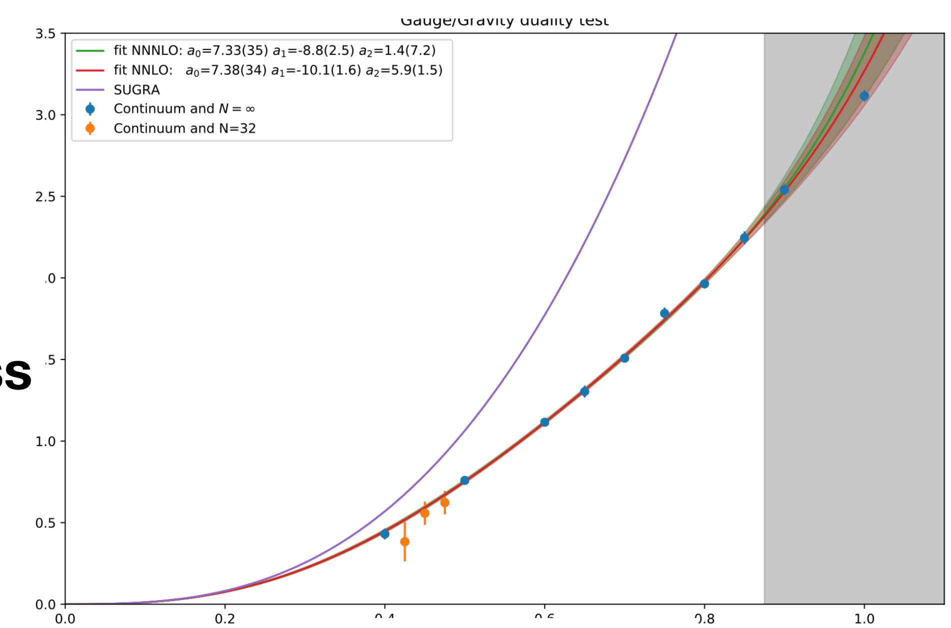
$$\begin{pmatrix} \psi^{11} \ldots\ldots \psi^{1N} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \psi^{N1} \ldots\ldots \psi^{NN} \end{pmatrix}$$

(p+1)-d maximal super Yang-Mills = black p-brane

(Itzhaki-Maldacena-Sonnenschein-Yankielowicz, 1998)

black hole (p=0)

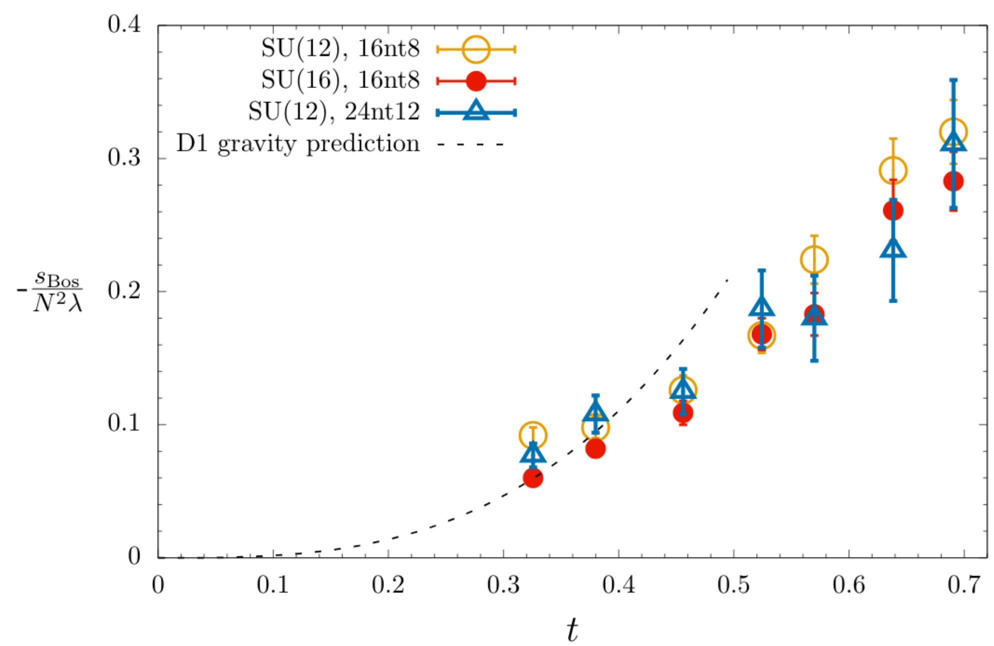
energy = BH mass



Temperature

Monte Carlo String/M-theory Collaboration, 2017

black string (p=1)



Catterall-Jha-Schaich-Wiseman, 2017

- Only special theories (maximally supersymmetric etc) describe gravity/string theory.

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  weakly coupled string/gravity.

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- Various theories, including QCD, describe some (not necessarily weakly coupled) string theory.

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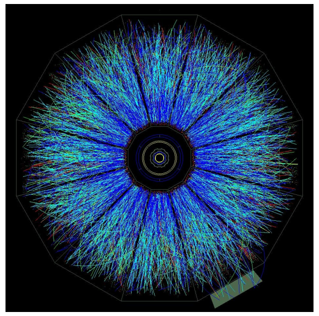
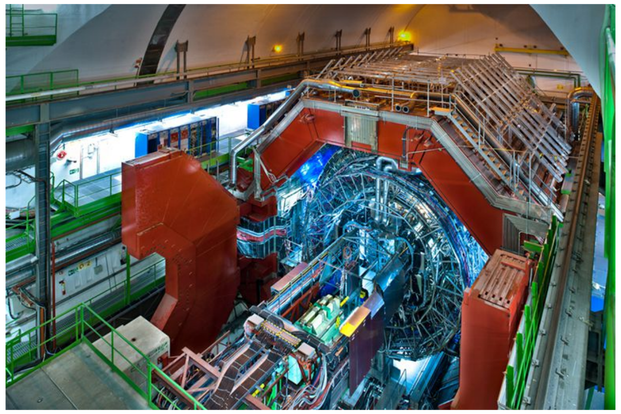
weakly coupled string/gravity.

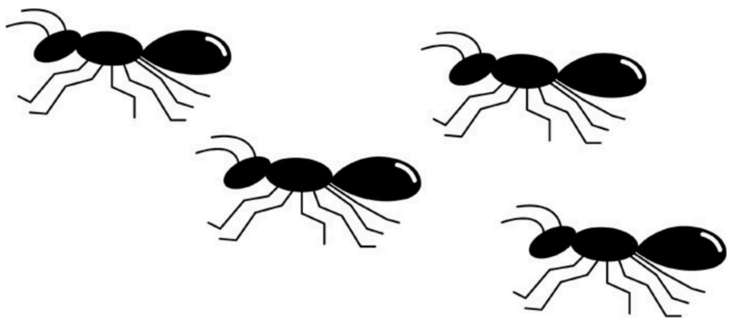
- Various theories, including QCD, describe some (not necessarily weakly coupled) string theory.

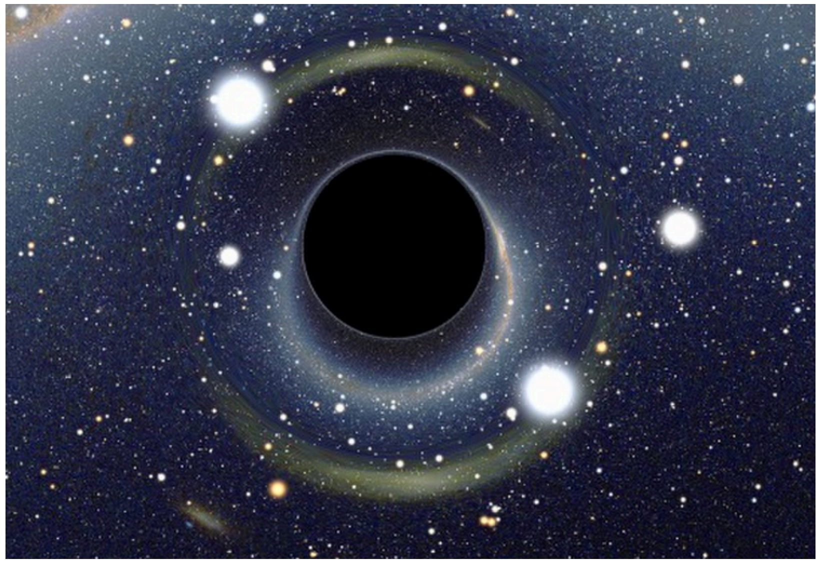
- Some ‘stringy’ features can be universal.



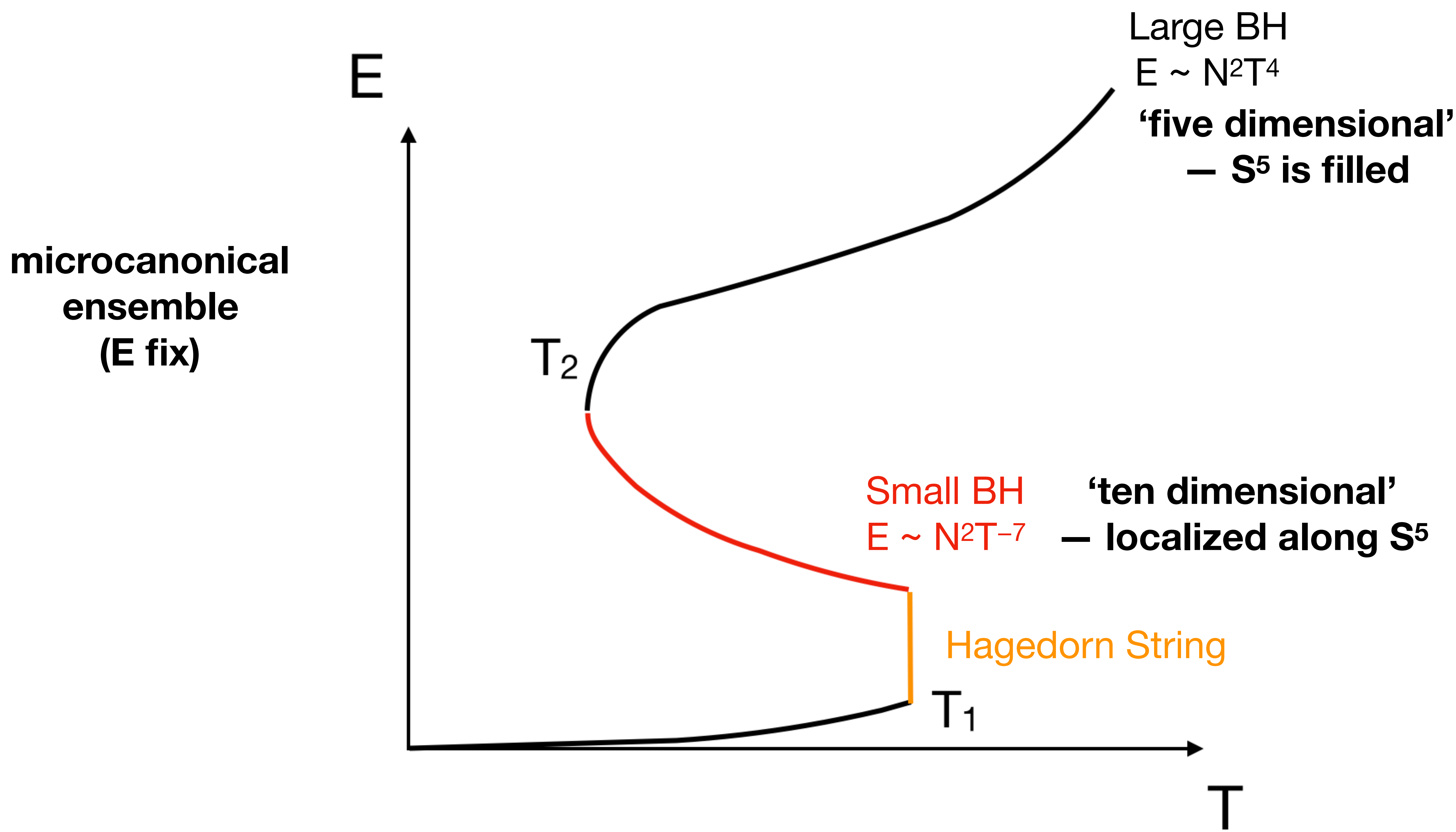


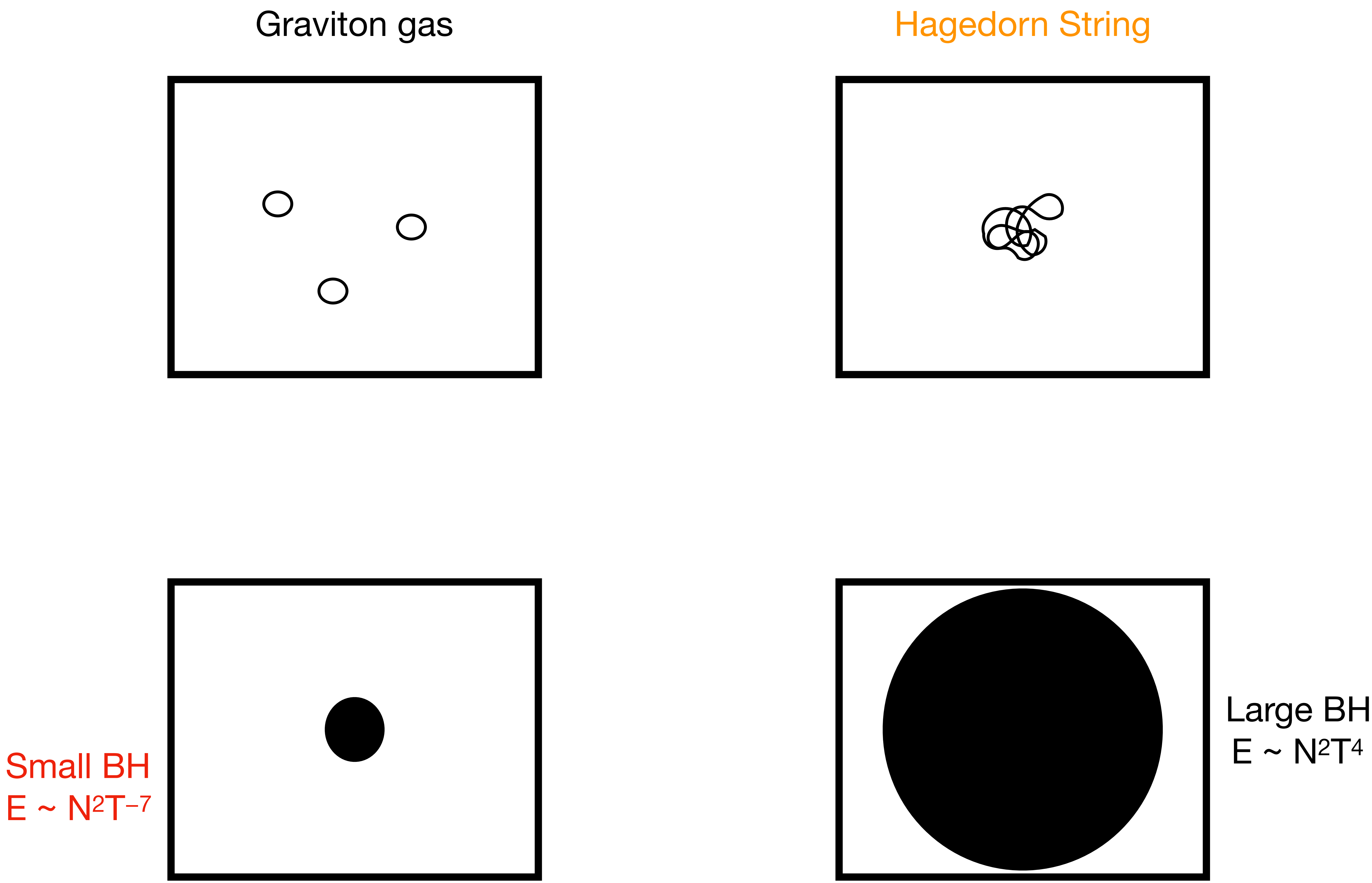
universal feature?



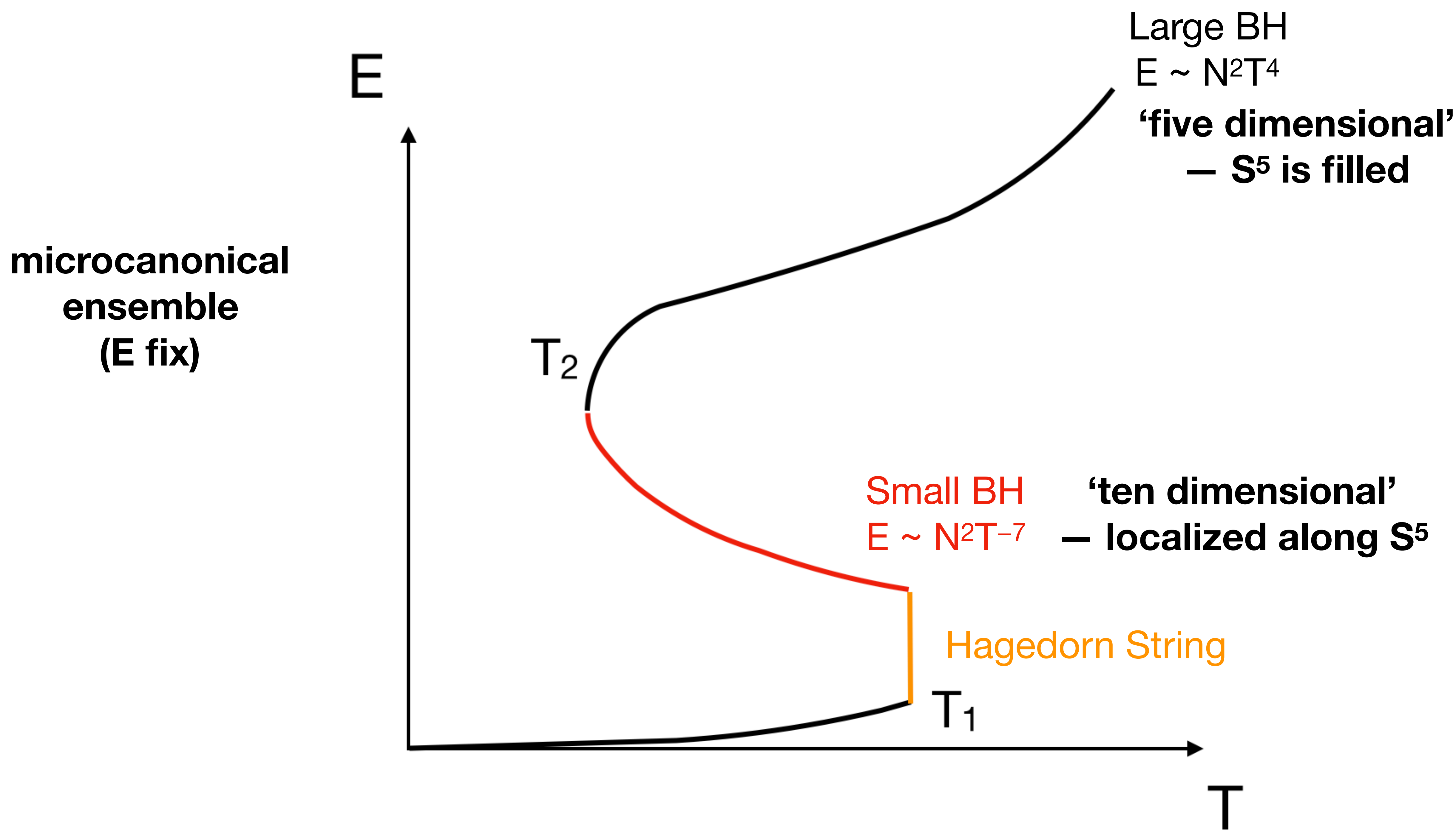


Black Hole in $AdS_5 \times S^5$ = 4d N=4 SYM on S^3



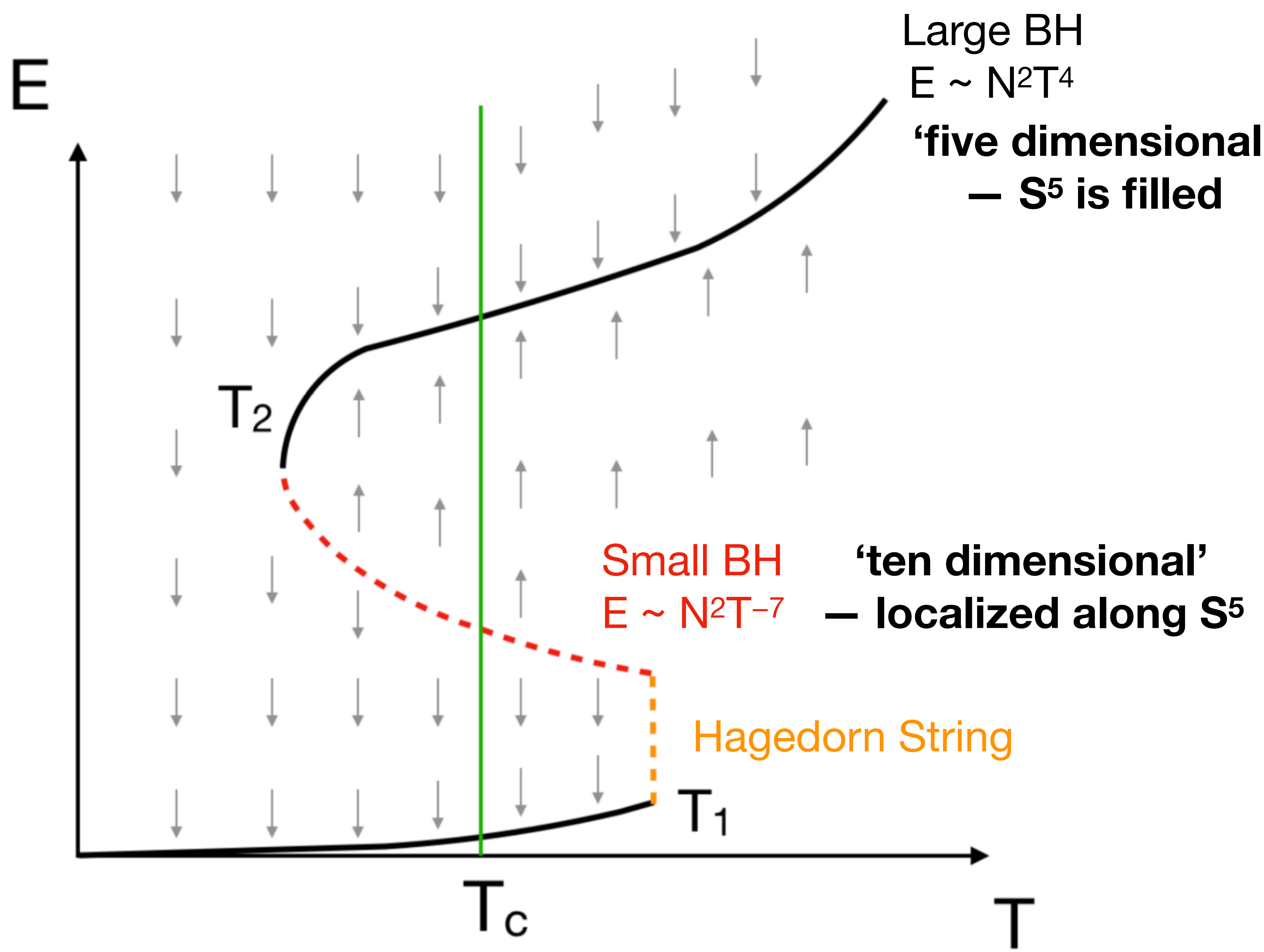

Graviton gas
Hagedorn String
Small BH
E ~ N²T⁻⁷
Large BH
E ~ N²T⁴

Black Hole in $AdS_5 \times S^5$ = 4d N=4 SYM on S^3

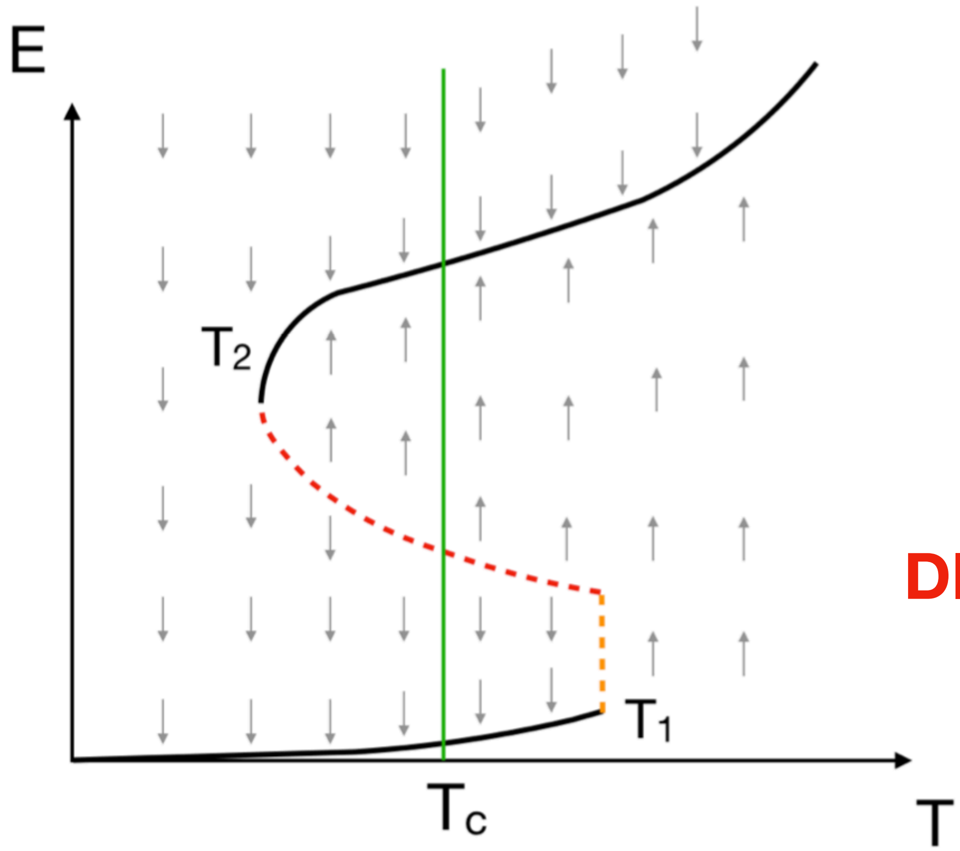


Black Hole in $AdS_5 \times S^5$ = 4d N=4 SYM on S^3

canonical ensemble (T fix)

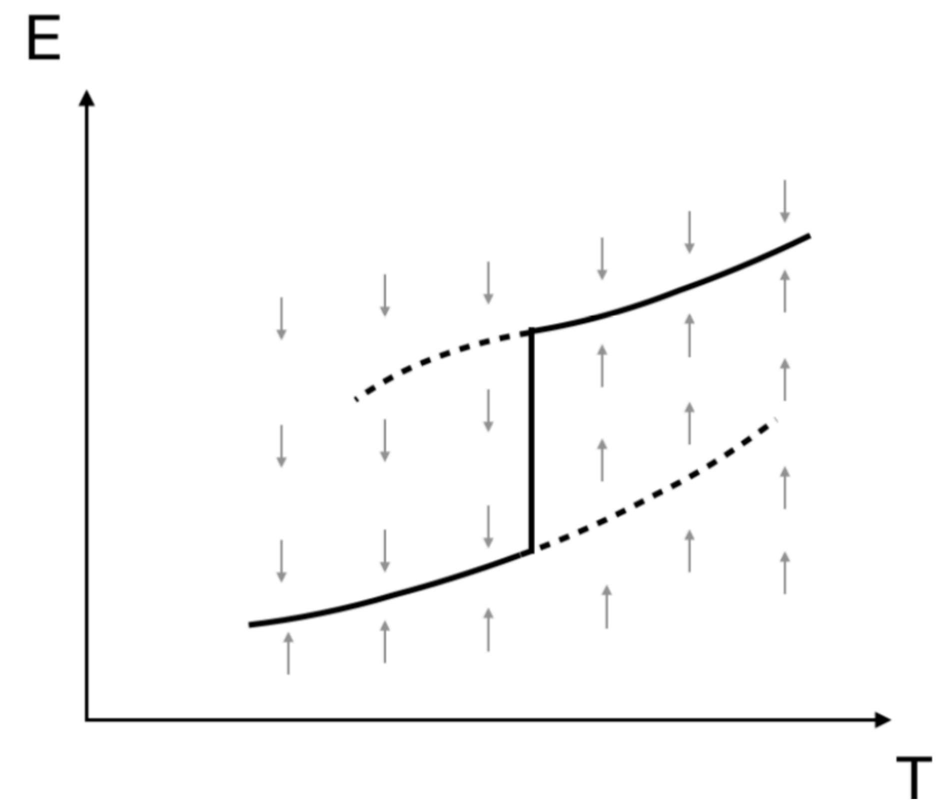


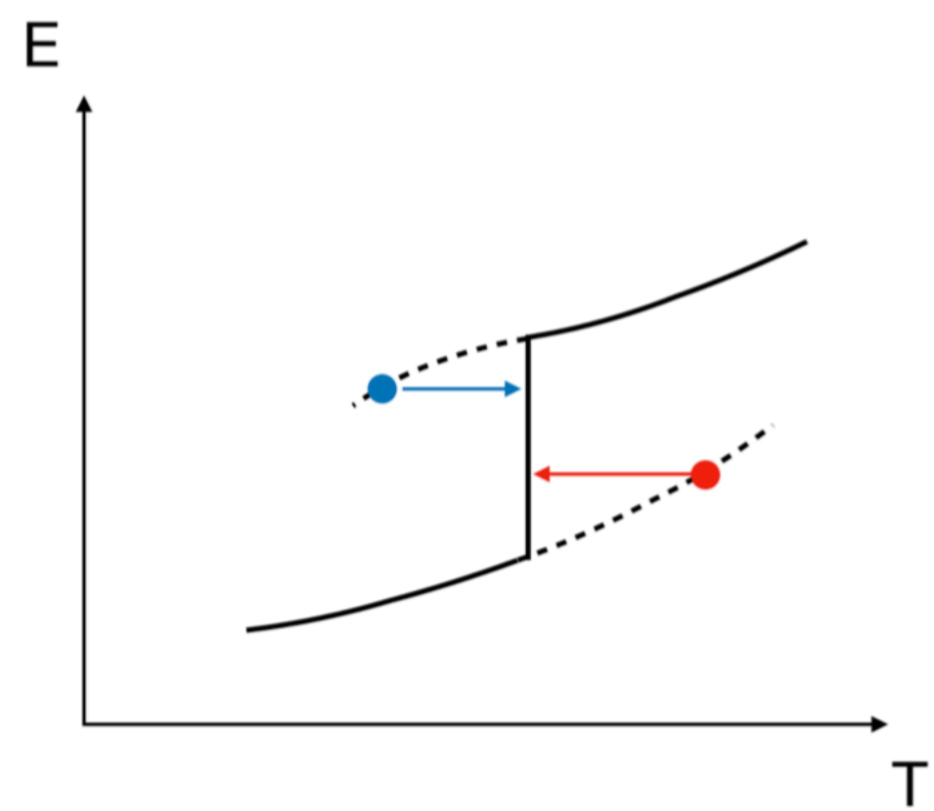
strongly coupled 4d SYM



VERY DIFFERENT

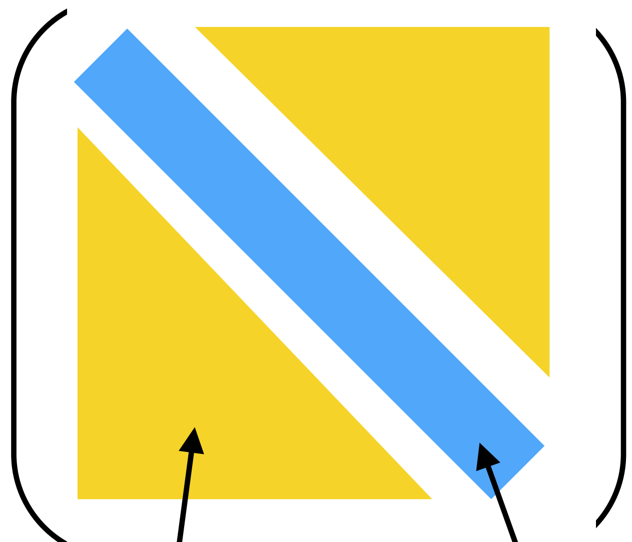
water/ice

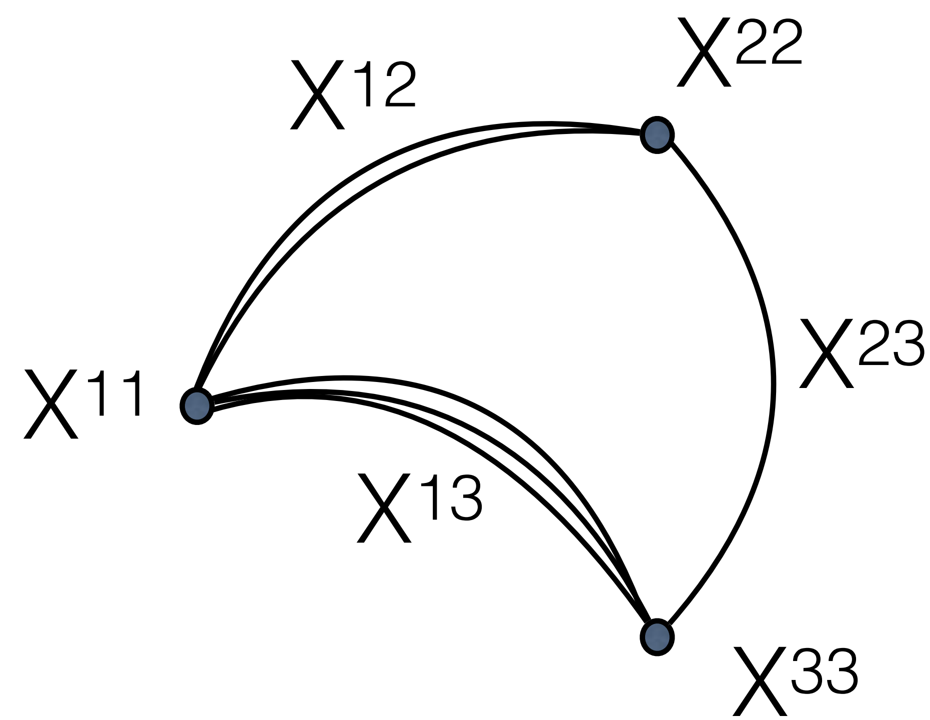




How can we explain such difference?

D-brane bound state and Gauge Theory

$X_M =$ 



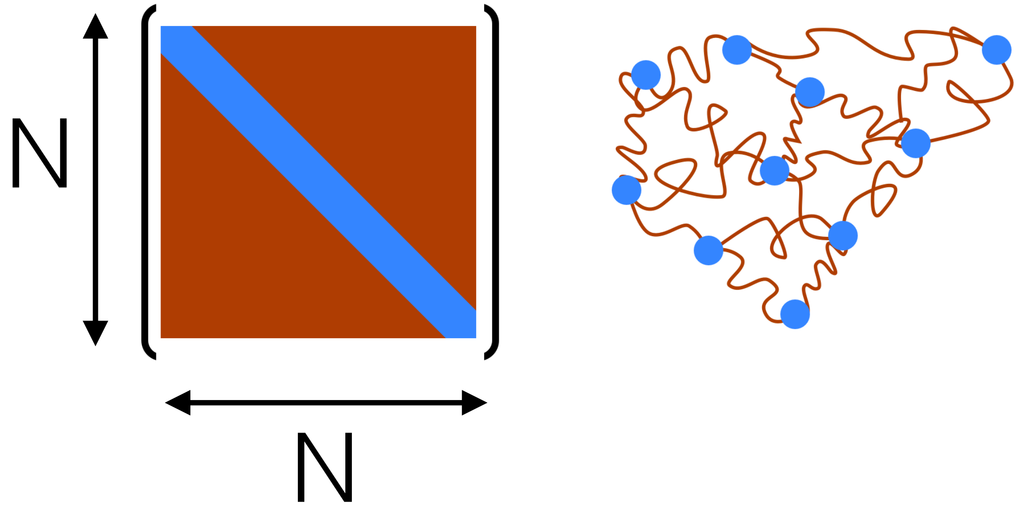
$(X_1^{ii}, X_2^{ii}, \ldots, X_6^{ii})$

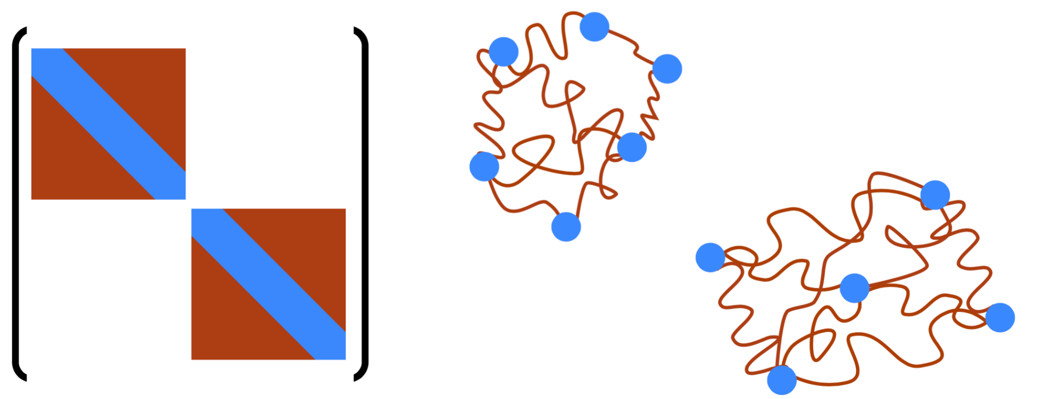
location of i-th D-brane

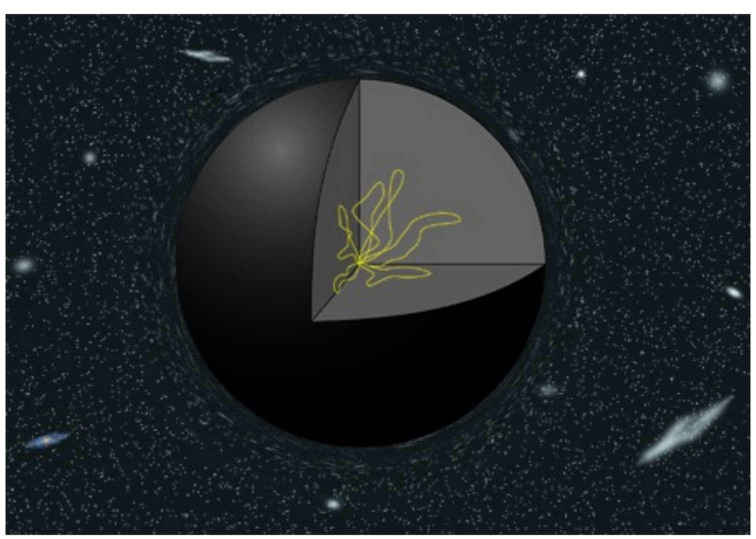
X_M^{ij} : open strings connecting i-th and j-th D-branes.

large value → a lot of strings are excited

(Witten, 1994)





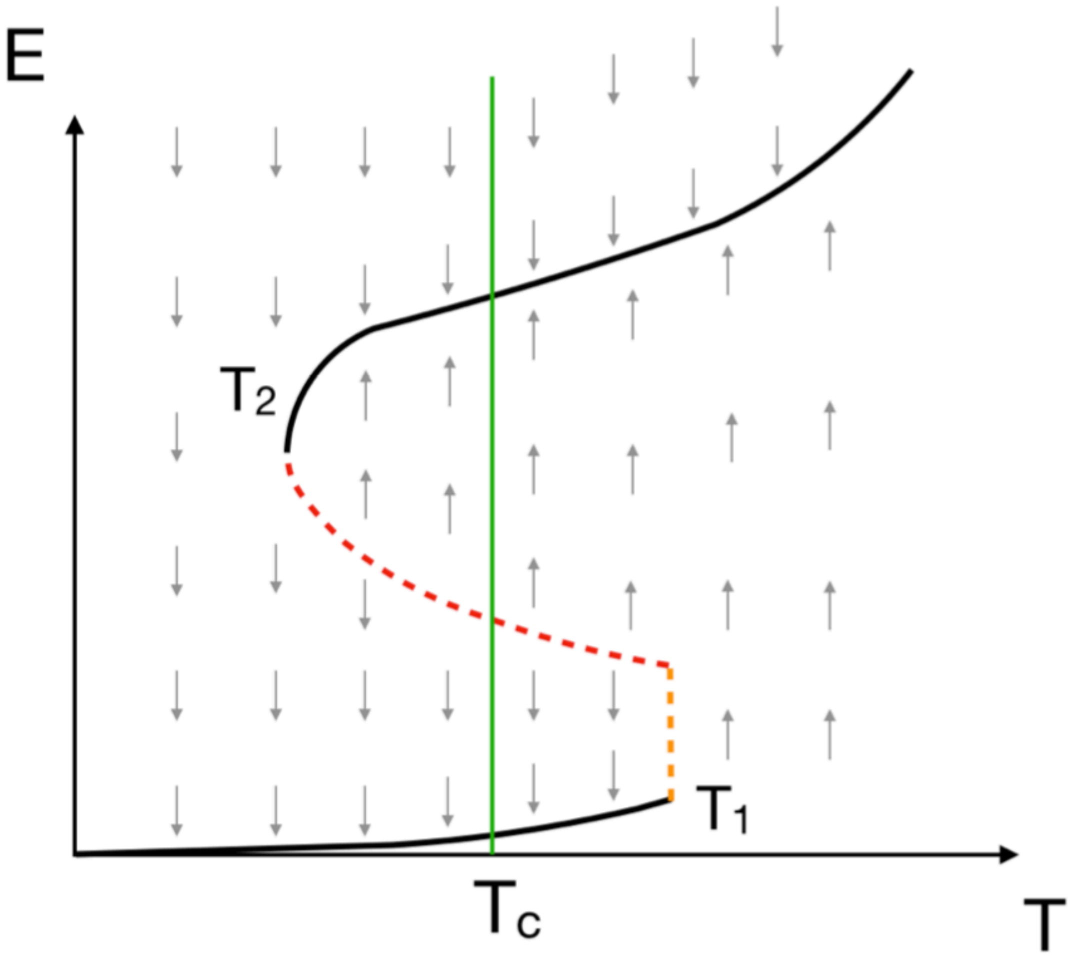


diagonal elements = particles (D-branes)
off-diagonal elements = open strings

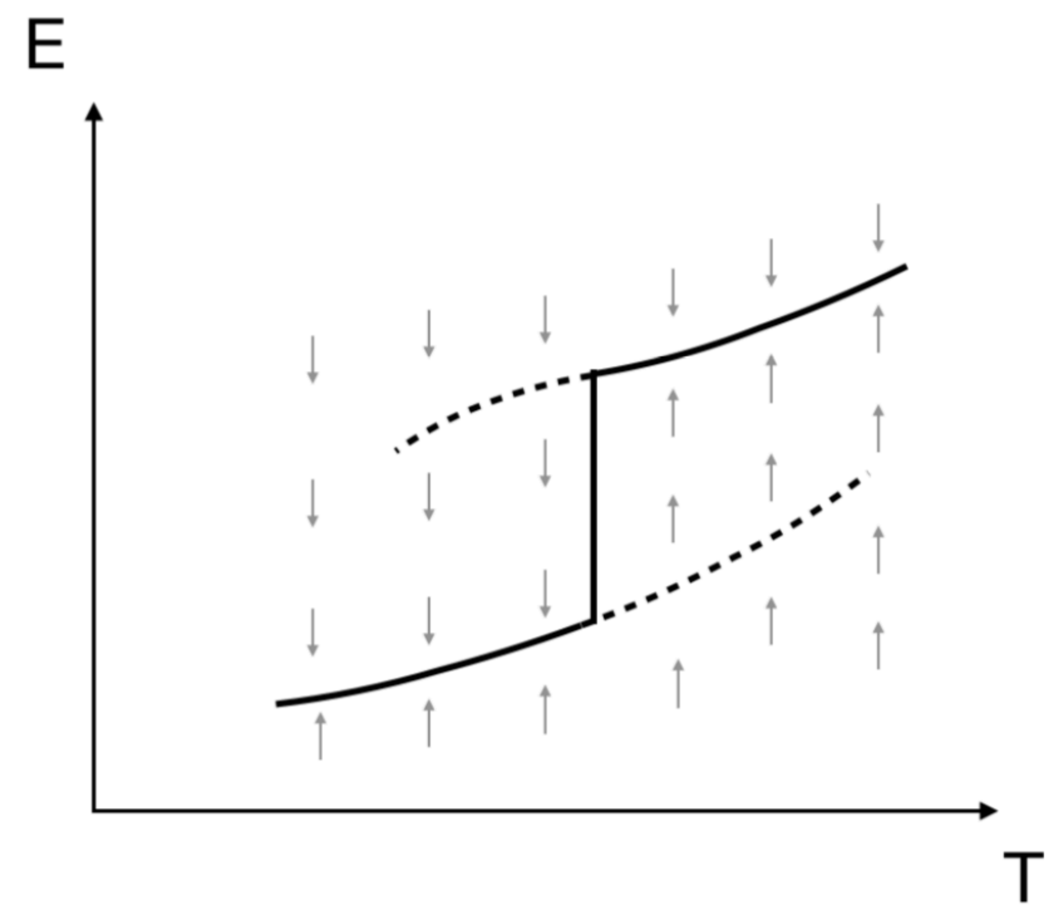
(Witten, 1994)

$\boxed{\text{black hole = bound state of D-branes and strings}}$

strongly coupled
4d SYM

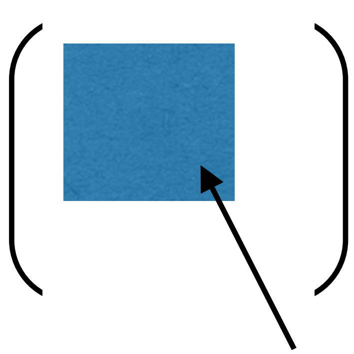


water/ice



VERY DIFFERENT

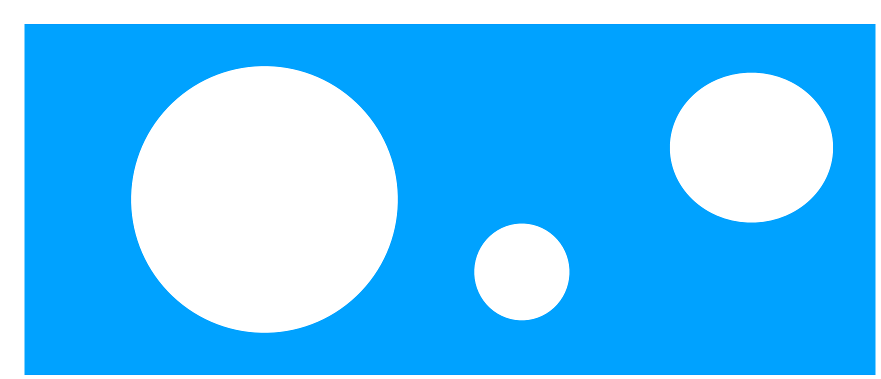
separation in color d.o.f

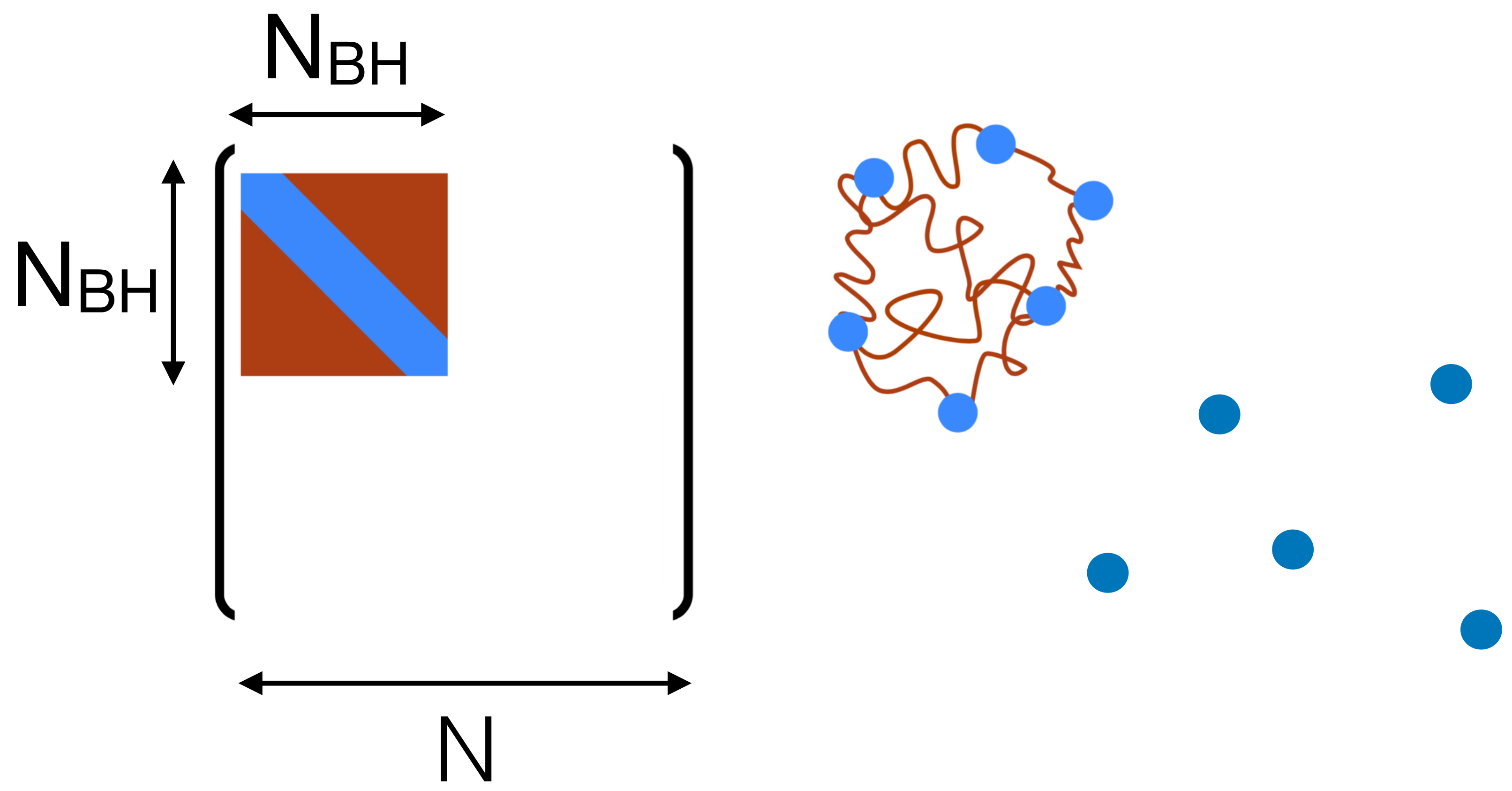


partially deconfine

(MH-Malts, 2016)

separation in space





N_{BH} D-branes form the bound state

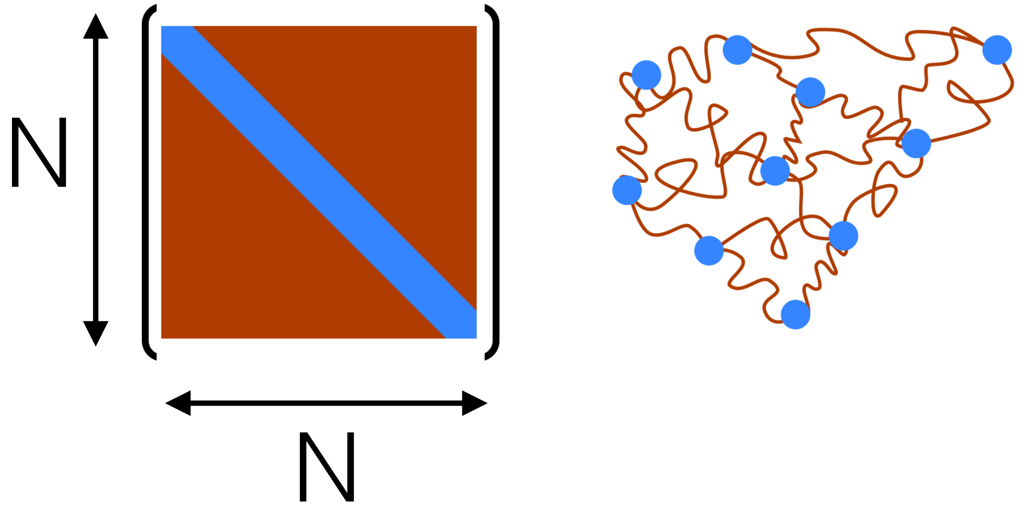
$U(N_{BH})$ is deconfined — ‘partial deconfinement’

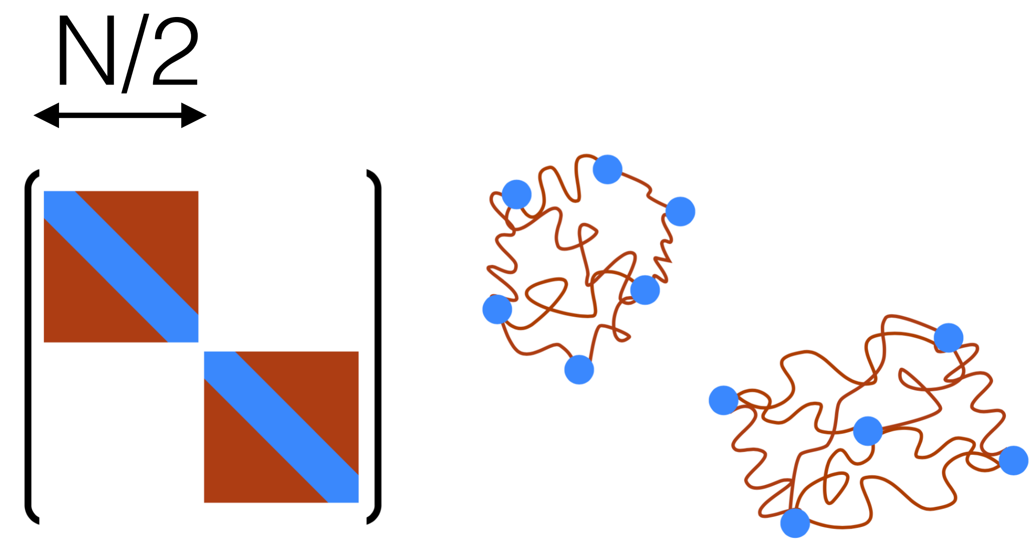
Can explain $E \sim N^2 T^{-7}$ for 4d SYM, $N^{3/2} T^{-8}$ for ABJM

(String Theory → 10d) (M-Theory → 11d)

(MH-Maltz, 2016)

Why can negative specific heat appear?



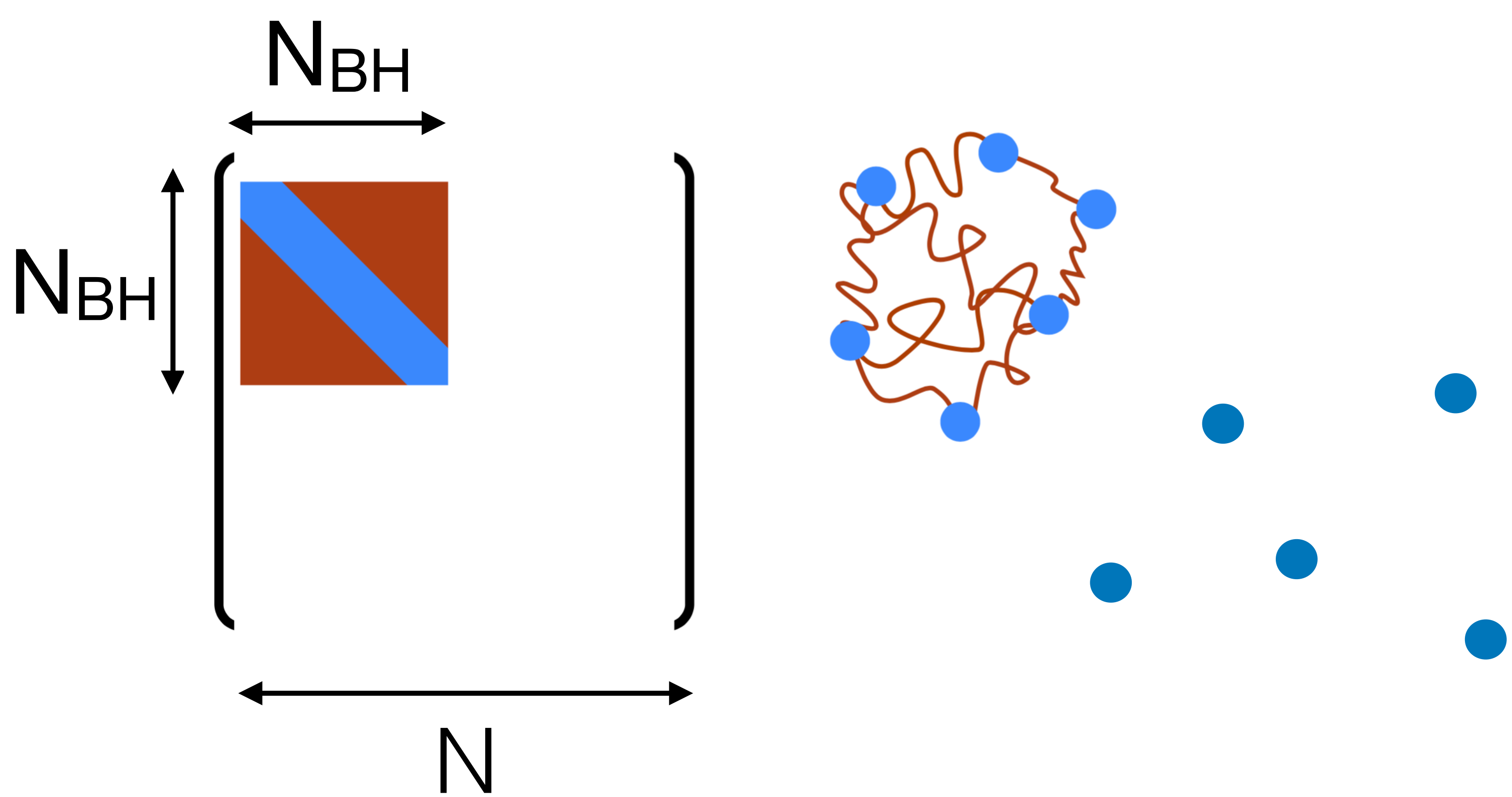


$T \sim E/N^2$

$T' \sim E'/[2\times(N/2)^2]$

$T' > T$ if $E' > E/2$

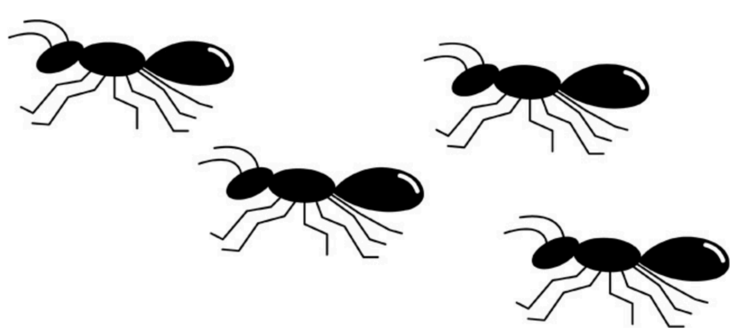
Why can negative specific heat appear?

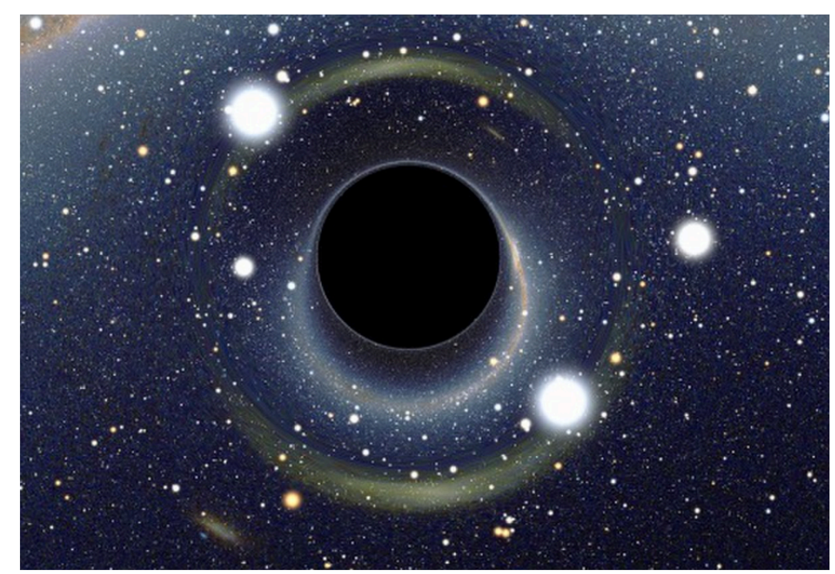


$$\boxed{T \sim E_{BH}/(N_{BH})^2}$$

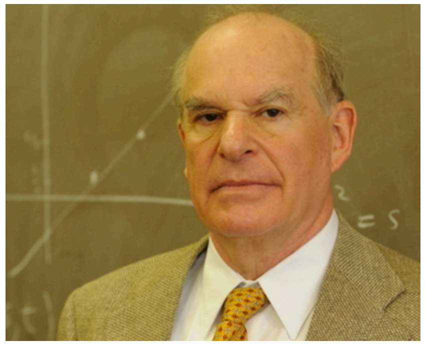
(more analyses later, or during coffee breaks)

Ant trail/black hole correspondence





MH-Ishiki-Watanabe, arXiv:1812.05494 [hep-th]



50th Anniversary

John H. Schwarz

California Institute of Technology

Strings 2018 June 29, 2018

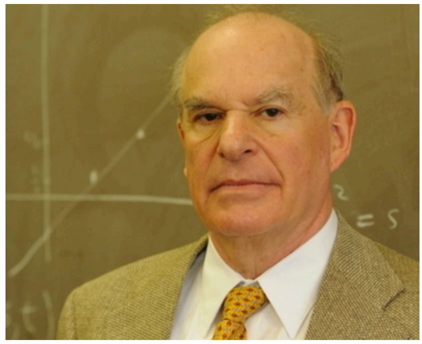
Lesson #2: Take "coincidences" seriously.

Example 1: The massless states of type IIA superstring theory correspond to the massless states of 11d supergravity on a circle. This was known for more than a decade before it was taken seriously.

M-theory (Witten)

Example 2: It was well known that the Lorentzian conformal group in d dimensions is the same as the Anti de Sitter isometry group in $d+1$ dimensions many years before AdS/CFT duality was proposed.

AdS/CFT (Maldacena)



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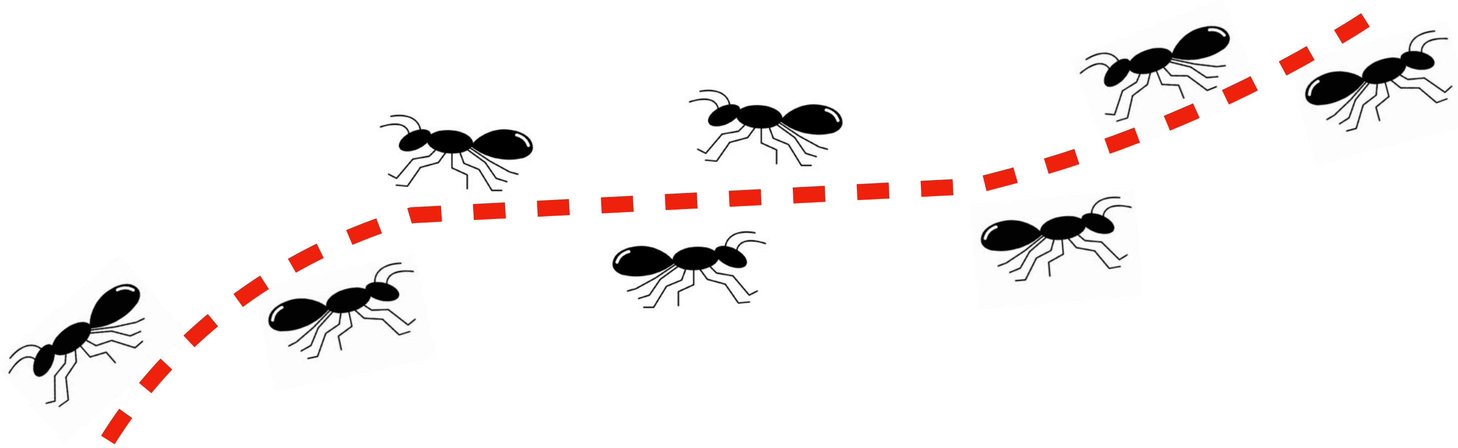
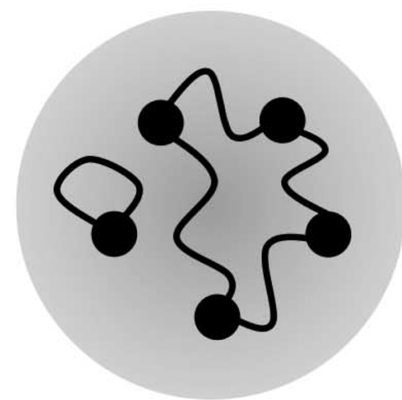


- Ant 'trail' is called 行列 in Japanese.
- 'Matrix' is called 行列 in Japanese.
- Gauge/gravity duality says BH is matrix.

$\boxed{\text{black hole = ant trail?}}$

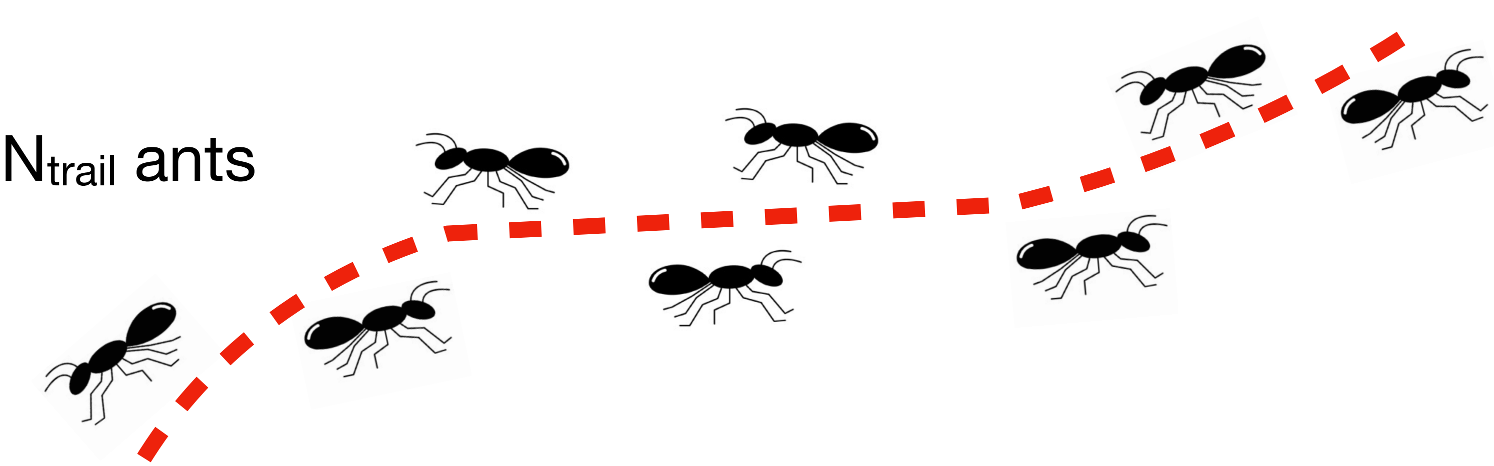
Black hole = D-brane bound by open strings

N_{BH} D-branes

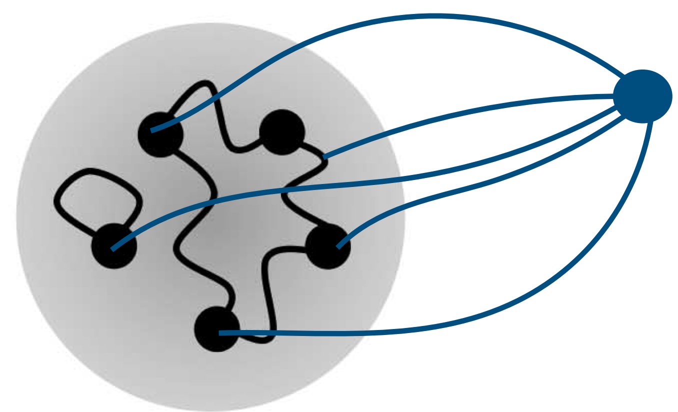


Ant trail = ants bound by pheromone

N_{trail} ants



Black hole = D-brane bound by open strings

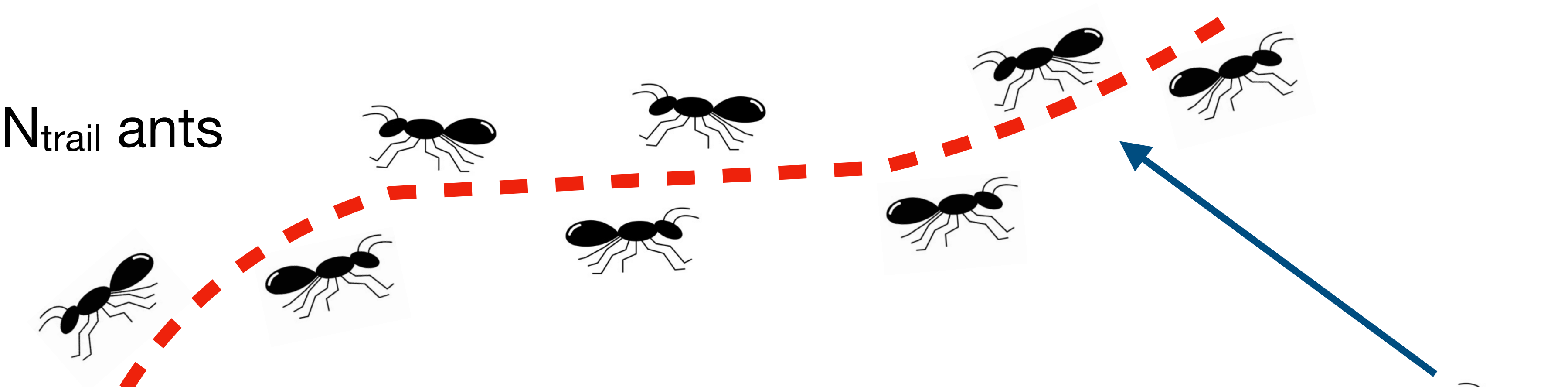


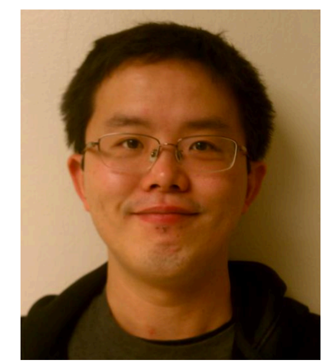
$$\boxed{N_{\mathrm{trail}} \longleftrightarrow N_{\mathrm{BH}}}$$

Ant trail = ants bound by pheromone

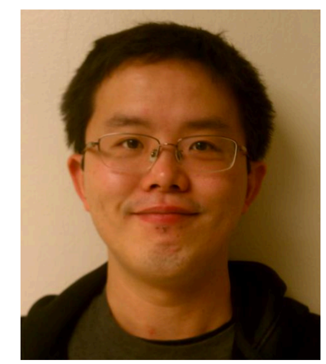
pheromone strength = p × N_{trail}

p: pheromone from each ant



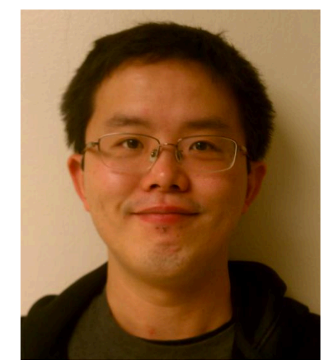


Lesson #2: Take "coincidences" seriously.



$$T_{\text{James}} > T_{\text{others}}$$

Lesson #2: Take “coincidences” seriously.



$$T_{\text{James}} > T_{\text{others}}$$

$$p_{\text{James}} > p_{\text{others}}$$

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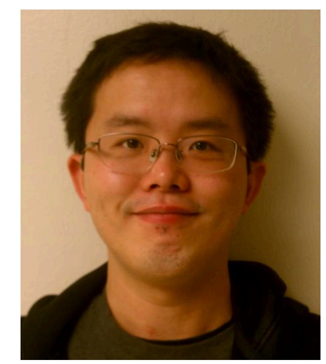


$$T_{\mathrm{James}} > T_{\mathrm{others}}$$

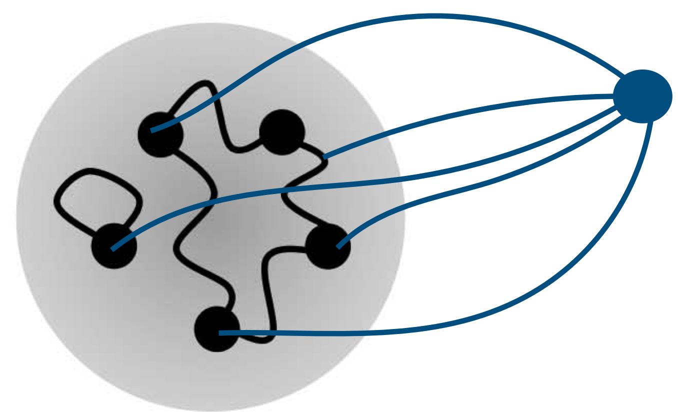
$$p_{\mathrm{James}} > p_{\mathrm{others}}$$

$$\boxed{T \sim p}$$

Lesson #2: Take “coincidences” seriously.

Black hole = D-brane bound by open strings

N_{BH} D-branes



N_{BH} open strings
try to capture
the other D-brane

$$\boxed{\begin{aligned} N_{\text{trail}} \quad &\longleftrightarrow \quad N_{\text{BH}}, \\ p \quad &\longleftrightarrow \quad T. \end{aligned}}$$

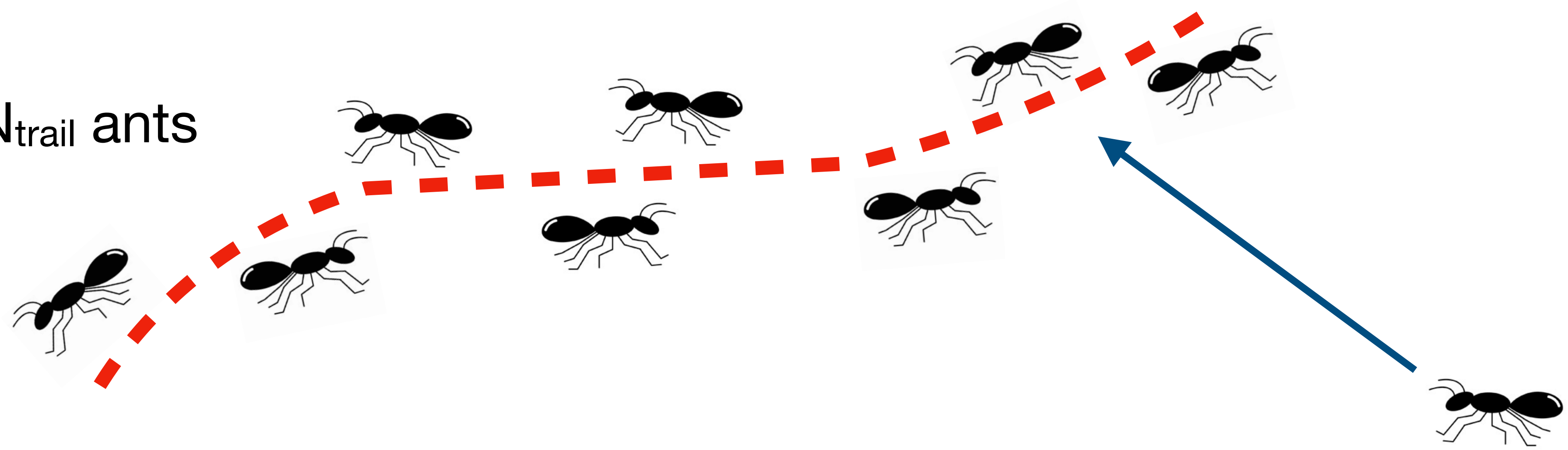
high T ∼ each mode is excited more
∼ stronger pheromone from each ant

Ant trail = ants bound by pheromone

pheromone strength = p × N_{trail}

p: pheromone from each ant

N_{trail} ants



The ant equation

Phase transition between disordered and ordered foraging in Pharaoh's ants

Madeleine Beekman*†, David J. T. Sumpter‡, and Francis L. W. Ratnieks*

*Laboratory of Apiculture and Social Insects, Department of Animal and Plant Sciences, Sheffield University, Sheffield S10 2TN, United Kingdom; and ‡Centre for Mathematical Biology, Mathematical Institute, Oxford University, 24-29 St. Giles, Oxford OX1 3LB, United Kingdom

Communicated by I. Prigogine, Free University of Brussels, Brussels, Belgium, June 7, 2001 (received for review August 12, 2000)

PNAS Proceedings of the National Academy of Sciences of the United States of America

$$\frac{dN_{\text{trail}}}{dt} = (\text{ants coming into the trail}) - (\text{ants leaving the trail})$$

$$= (\alpha + \underbrace{pN_{\text{trail}}}_{\text{stringy term}})(N - N_{\text{trail}}) - \frac{sN_{\text{trail}}}{s + N_{\text{trail}}} \mathbf{= 0}$$

Natural large-N limit: $\alpha \sim N^0, p \sim N^0, s \sim N^1$

(many-ant limit)

The ant equation

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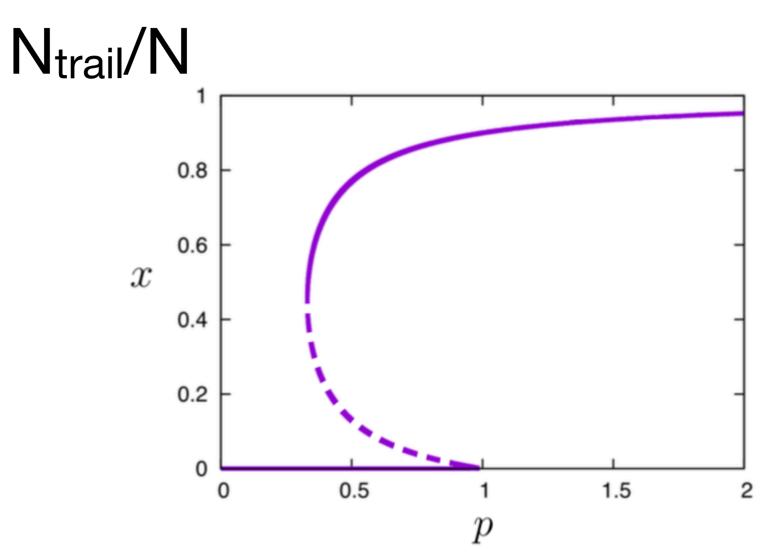
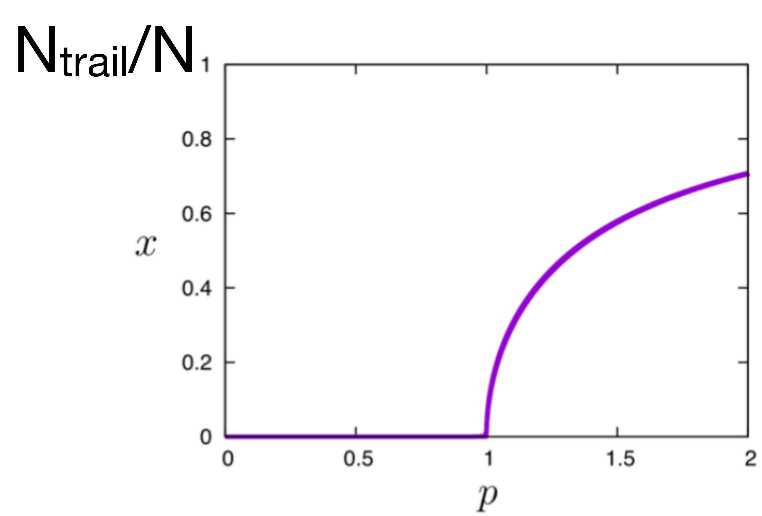
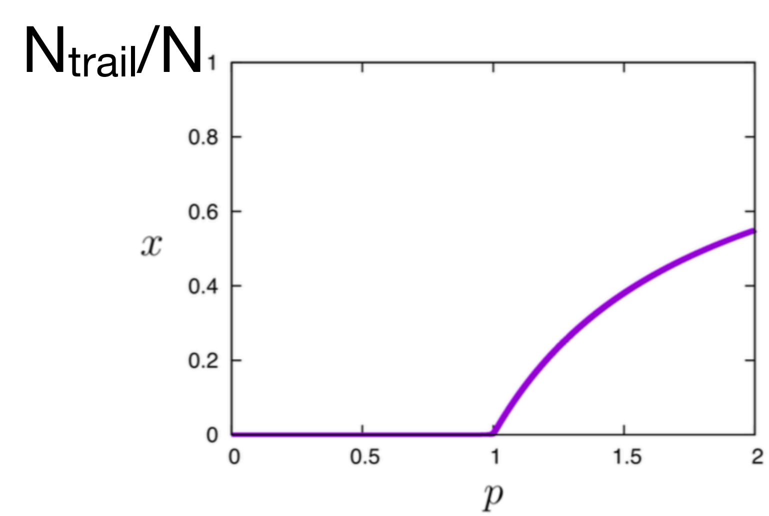
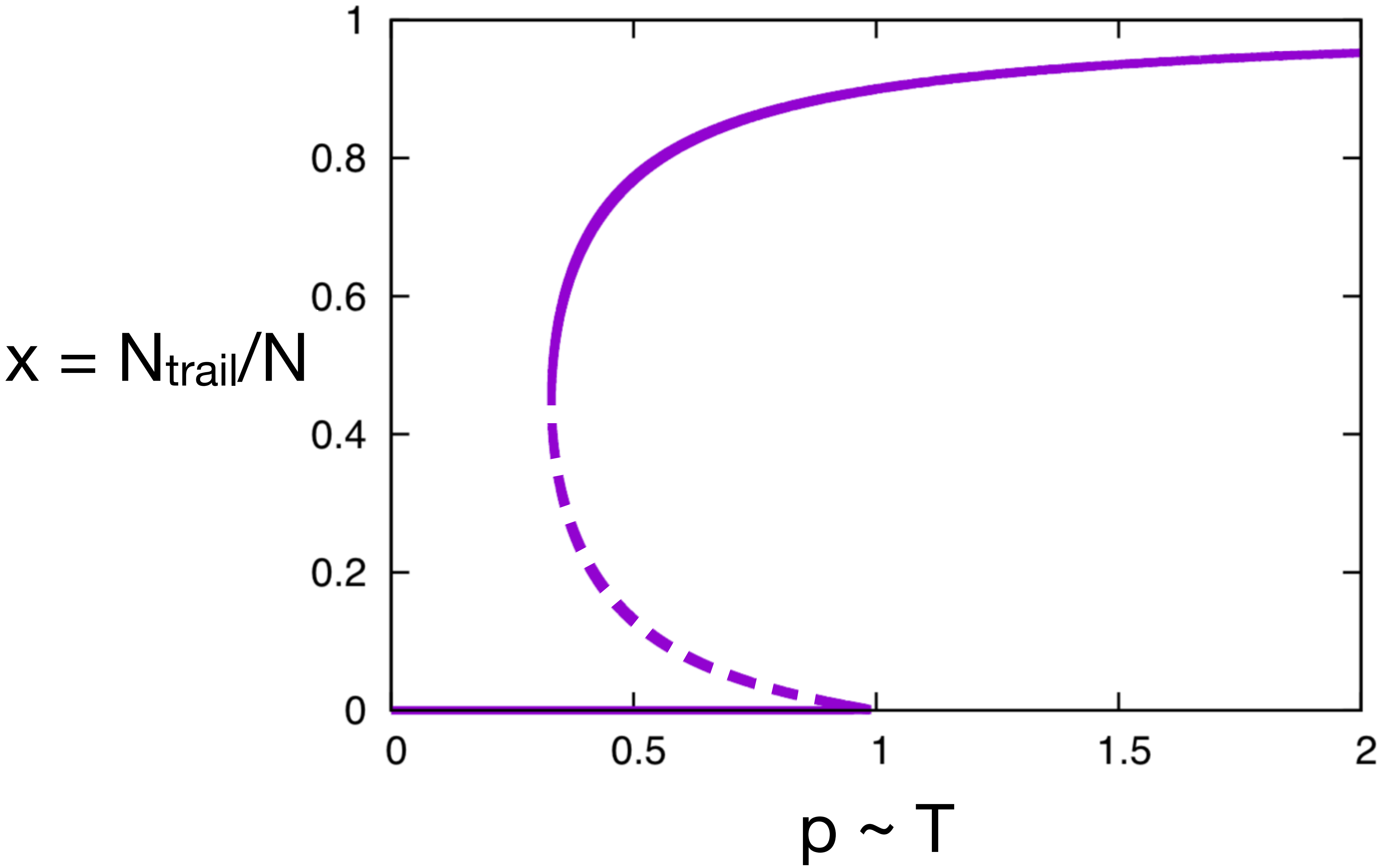
$$
\begin{aligned}
\frac{dN_{\mathrm{trail}}}{dt} &= (\text{ants coming into the trail}) - (\text{ants leaving the trail}) \\
&= (\alpha + \underbrace{pN_{\mathrm{trail}}}_{\text{stringy term}})(N - N_{\mathrm{trail}}) - \frac{sN_{\mathrm{trail}}}{s + N_{\mathrm{trail}}} \; \mathbf{= 0}
\end{aligned}
$$

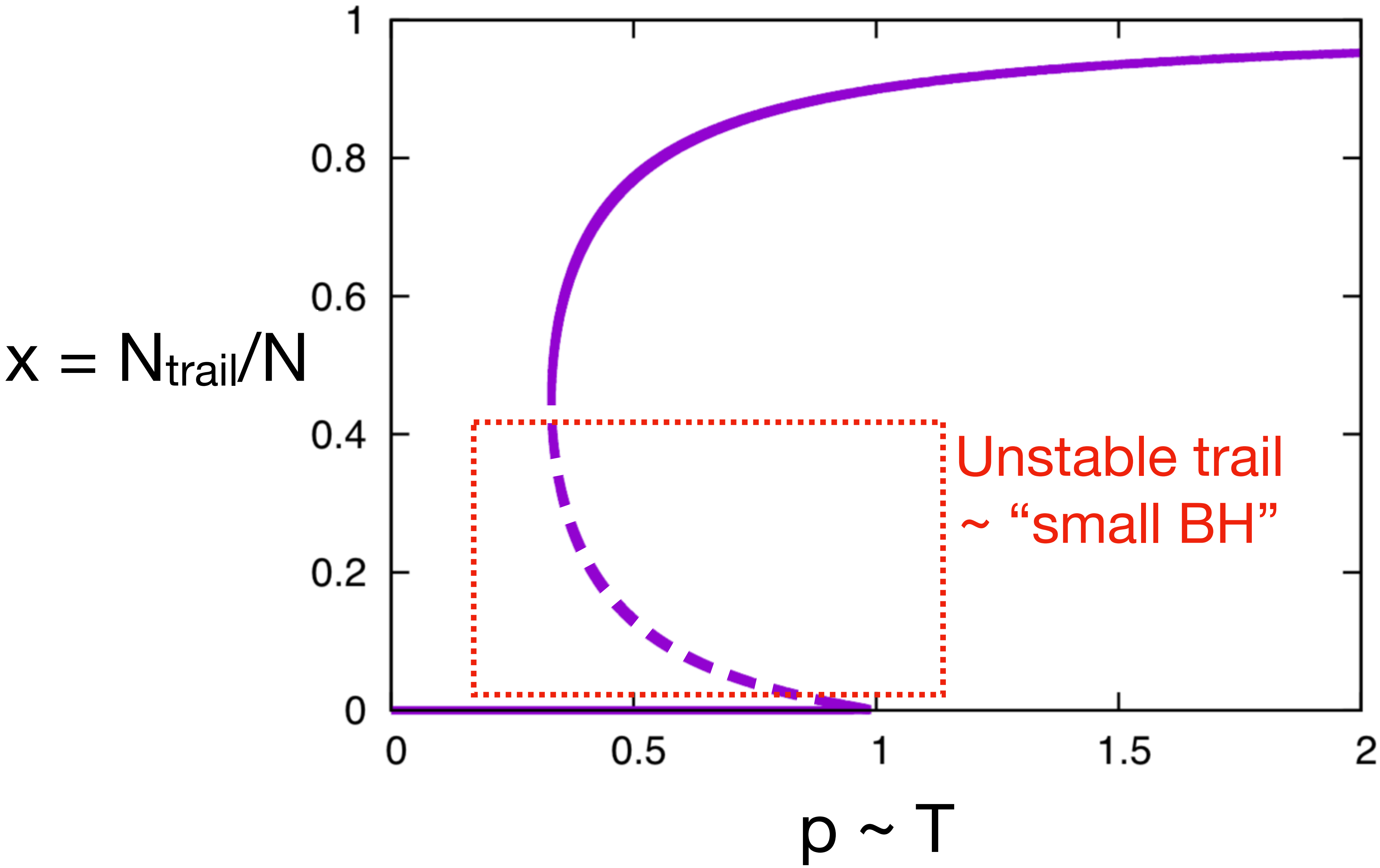




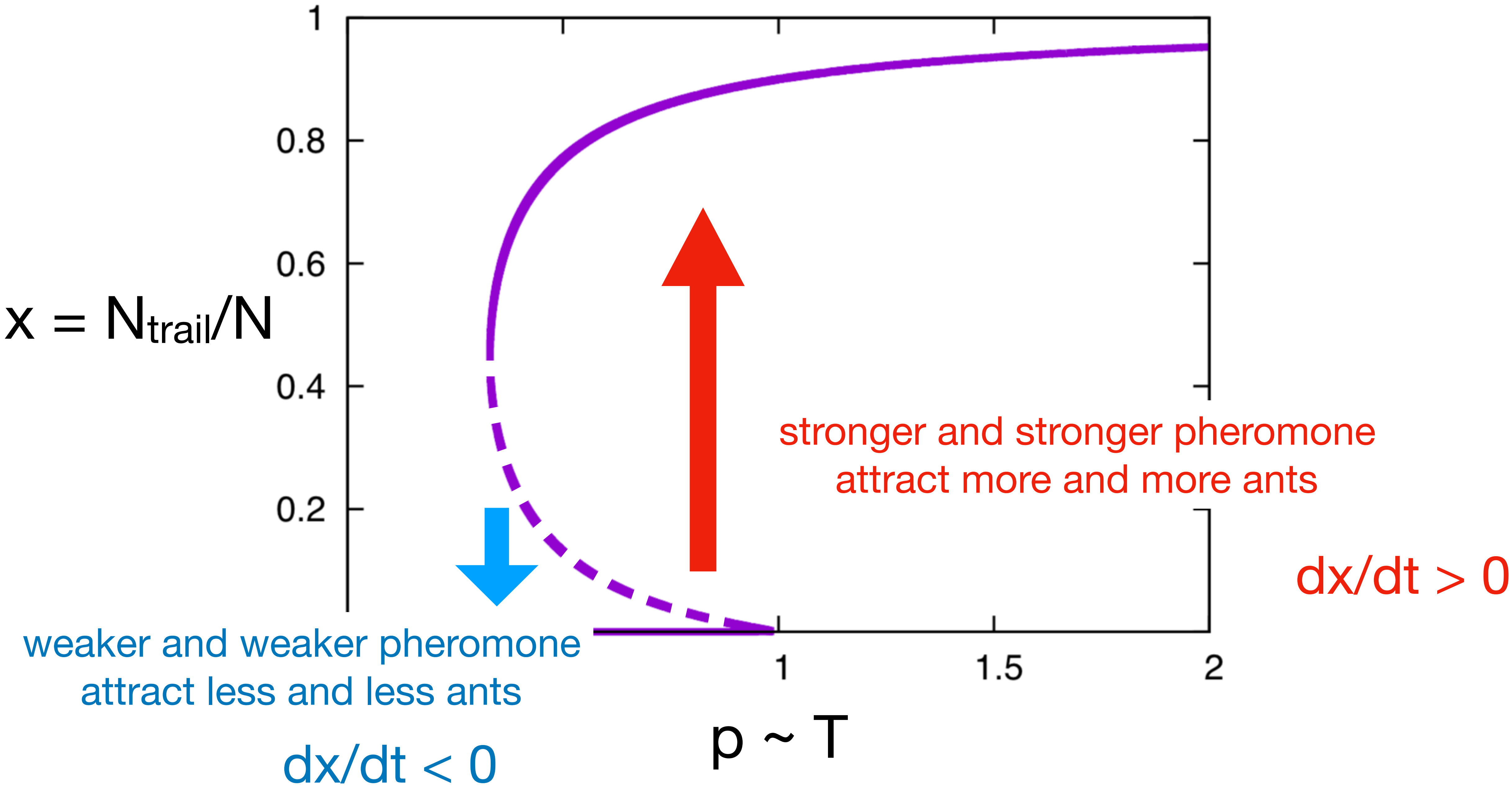




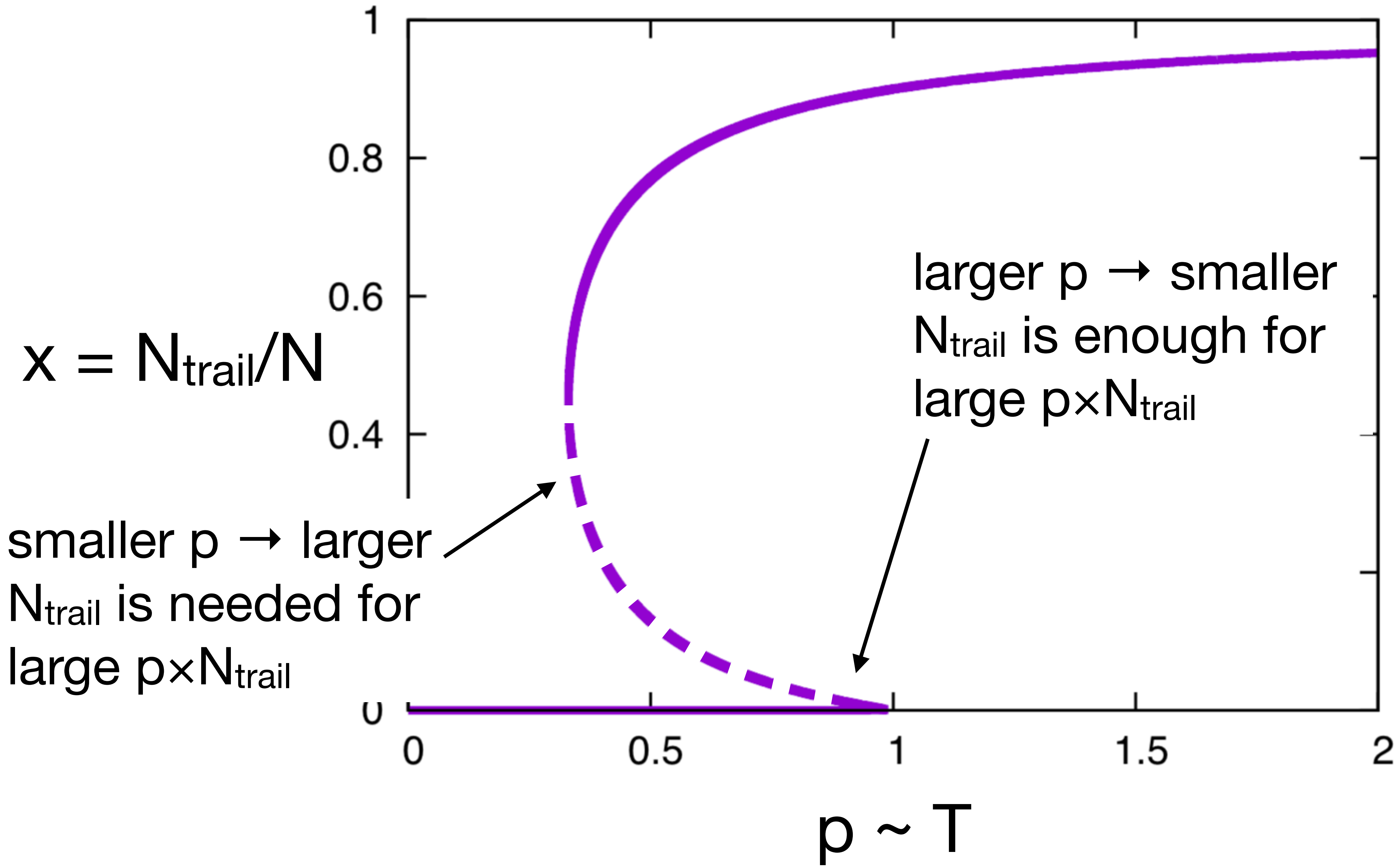
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
x = Ntrail/N
p ~ T



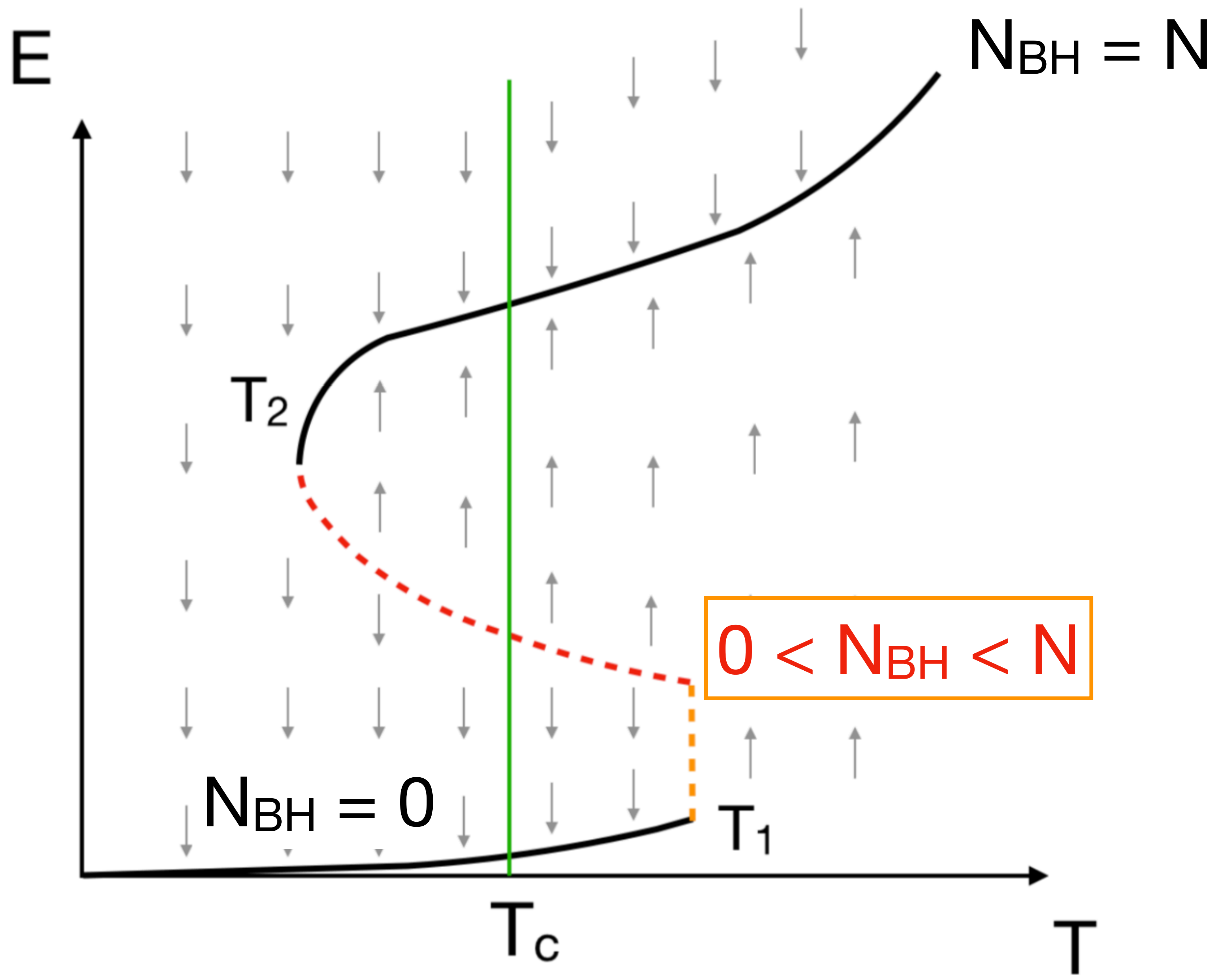
x = Ntrail/N
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
p ~ T
Unstable trail
~ “small BH”



1
0.8
0.6
0.4
0.2
x = N_{trail}/N
1
1.5
2
p ~ T
stronger and stronger pheromone
attract more and more ants
dx/dt > 0
weaker and weaker pheromone
attract less and less ants
dx/dt < 0

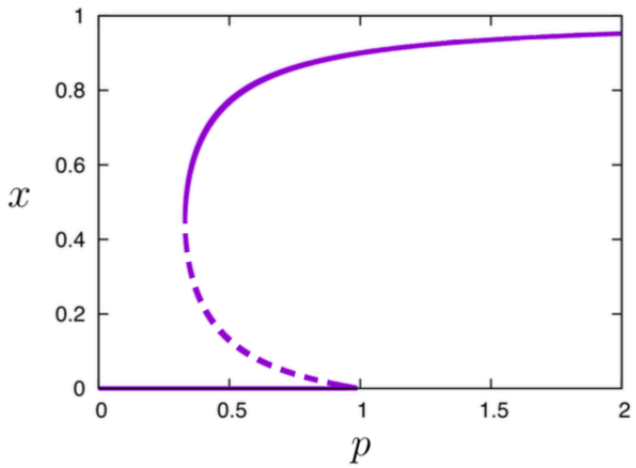


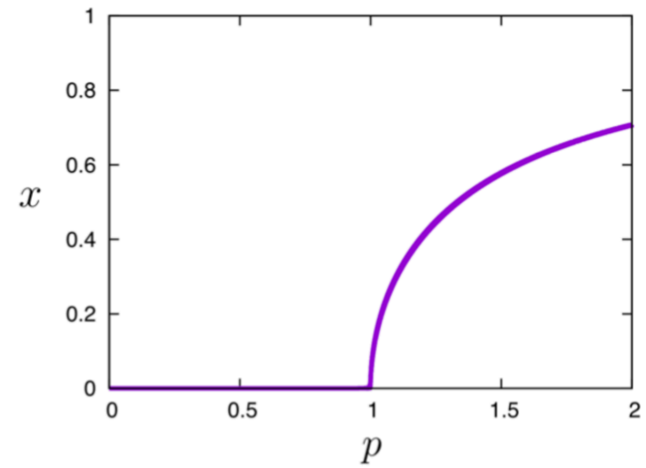
x = Ntrail/N
p ~ T
1
0.8
0.6
0.4
0
0
0.5
1
1.5
2
larger p → smaller
Ntrail is enough for
large p×Ntrail
smaller p → larger
Ntrail is needed for
large p×Ntrail

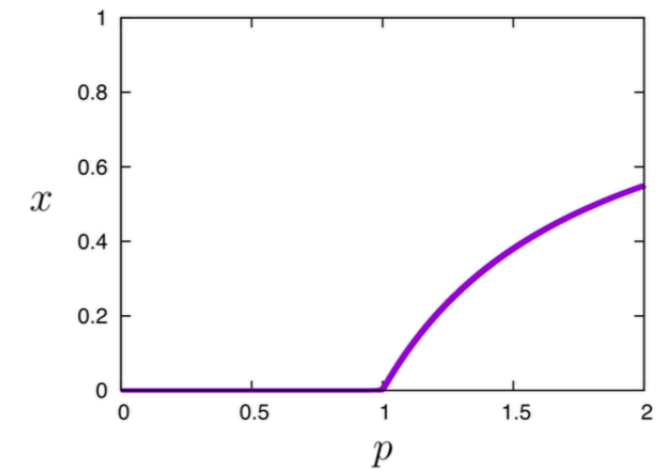


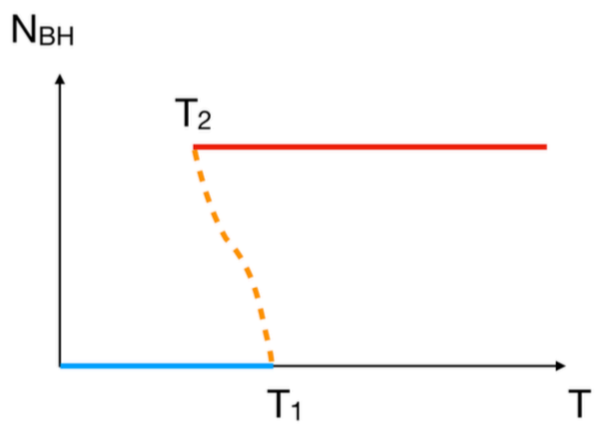
N_{BH} D-branes form the bound state

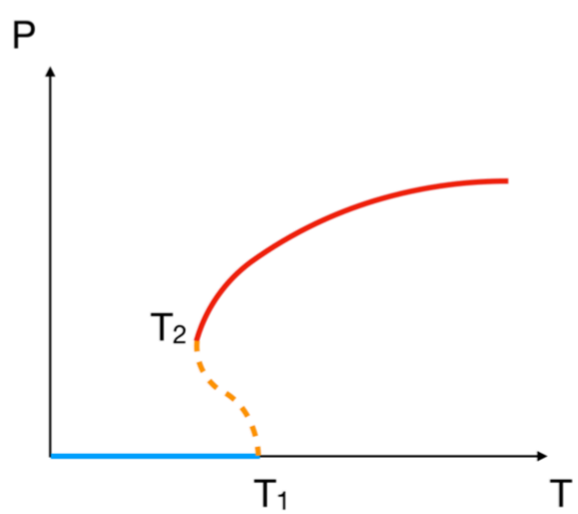
$U(N_{BH})$ is deconfined — 'partial deconfinement'

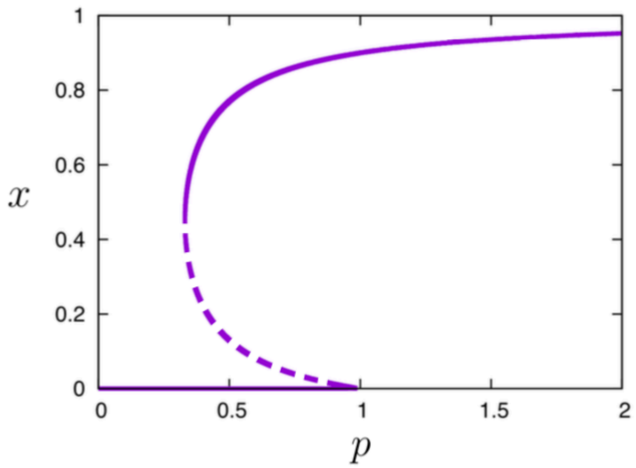

1
0.8
0.6
0.4
0.2
0
x
0
0.5
1
1.5
2
p

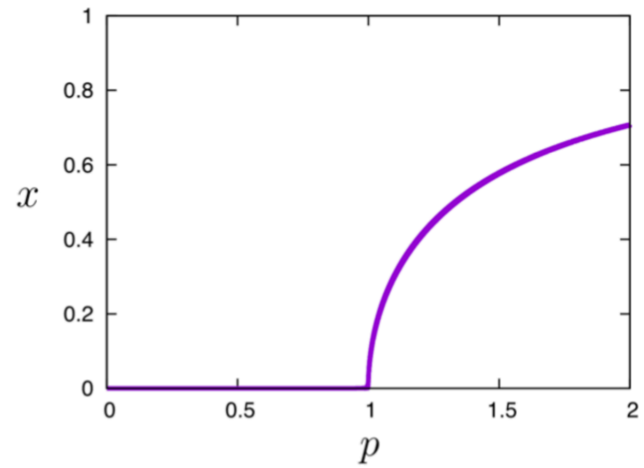

1
0.8
0.6
0.4
0.2
0
x
0
0.5
1
1.5
2
p

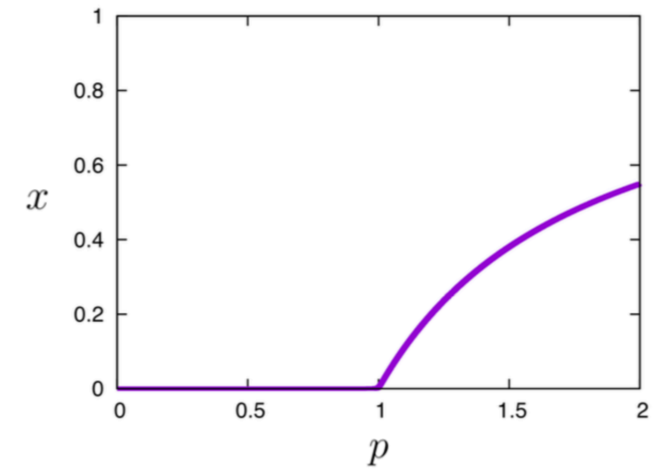

1
0.8
0.6
0.4
0.2
0
x
0
0.5
1
1.5
2
p


N_{BH}
T_2
T_1
T

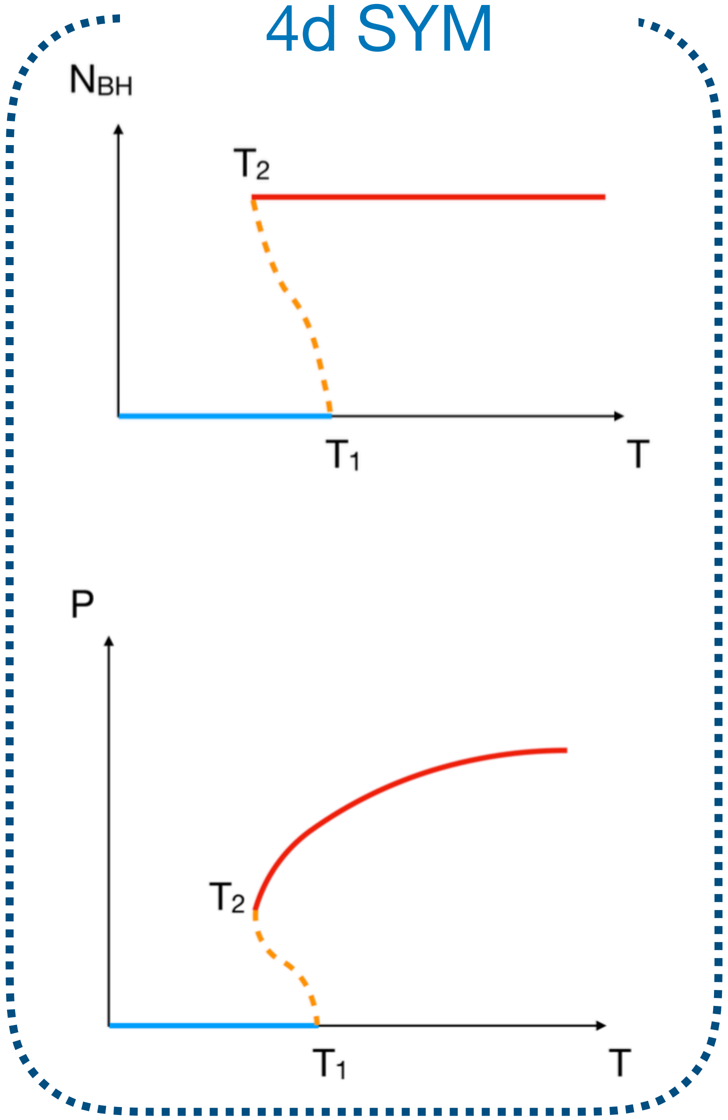

P
T_2
T_1
T

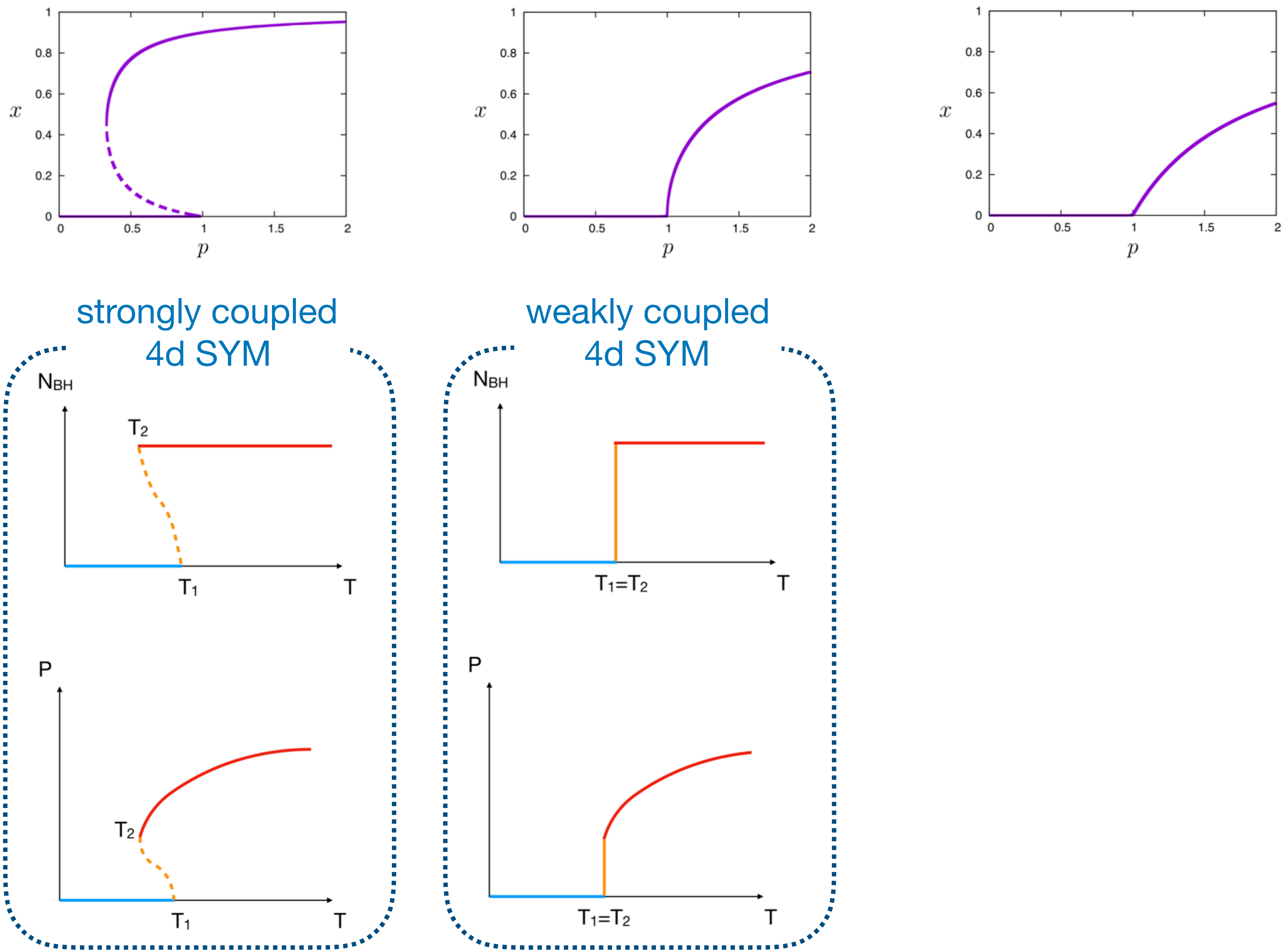




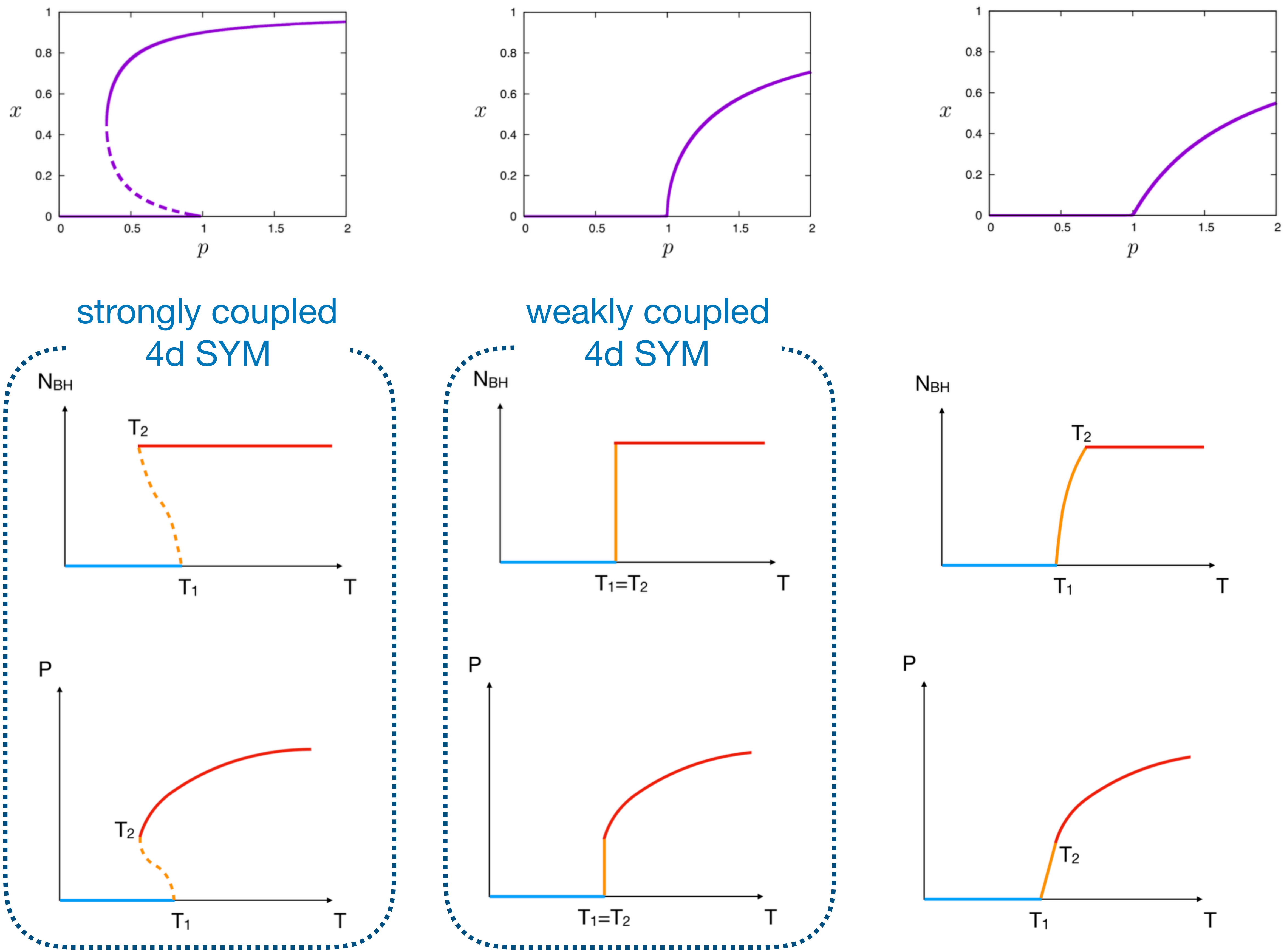


strongly coupled
4d SYM

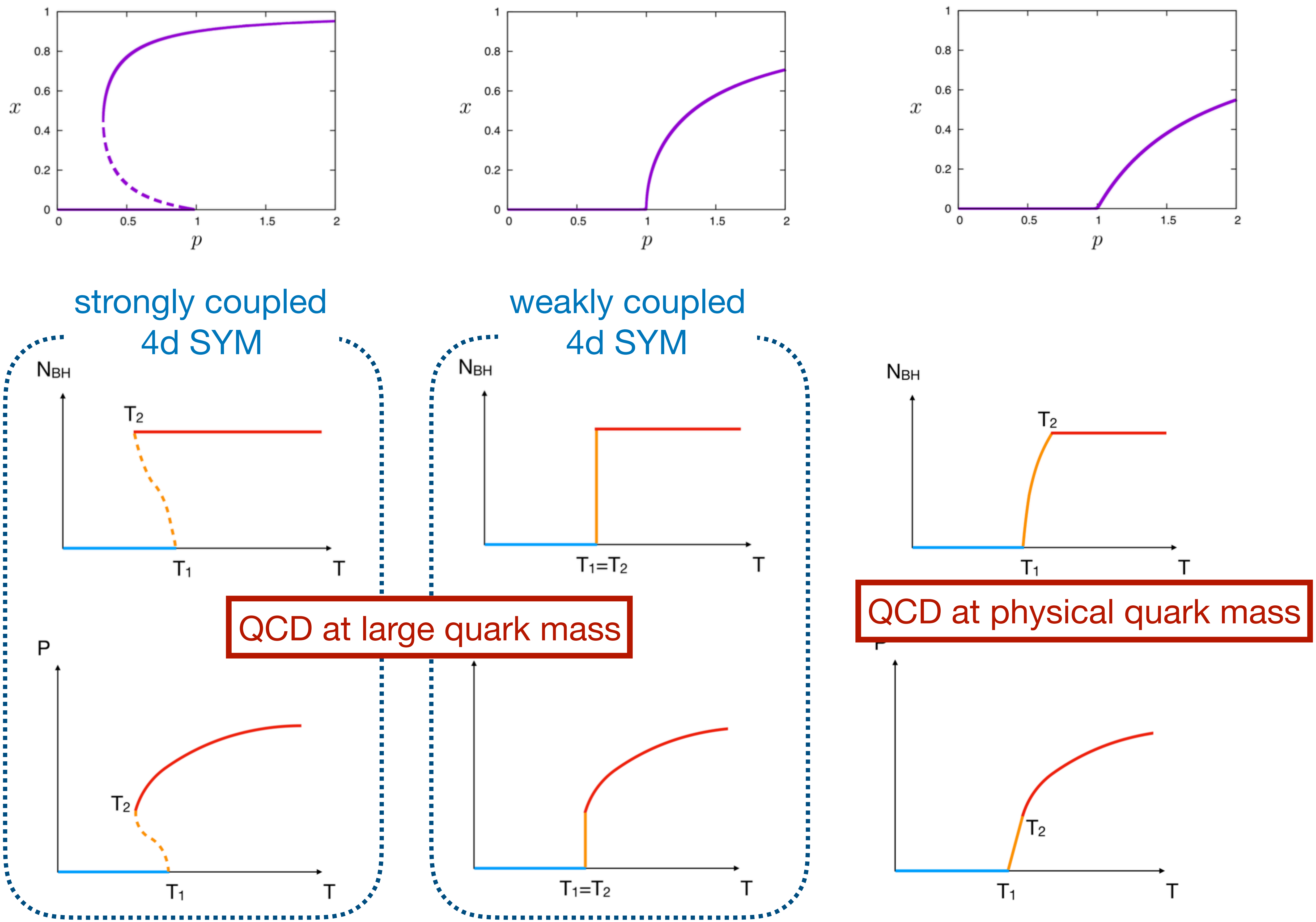




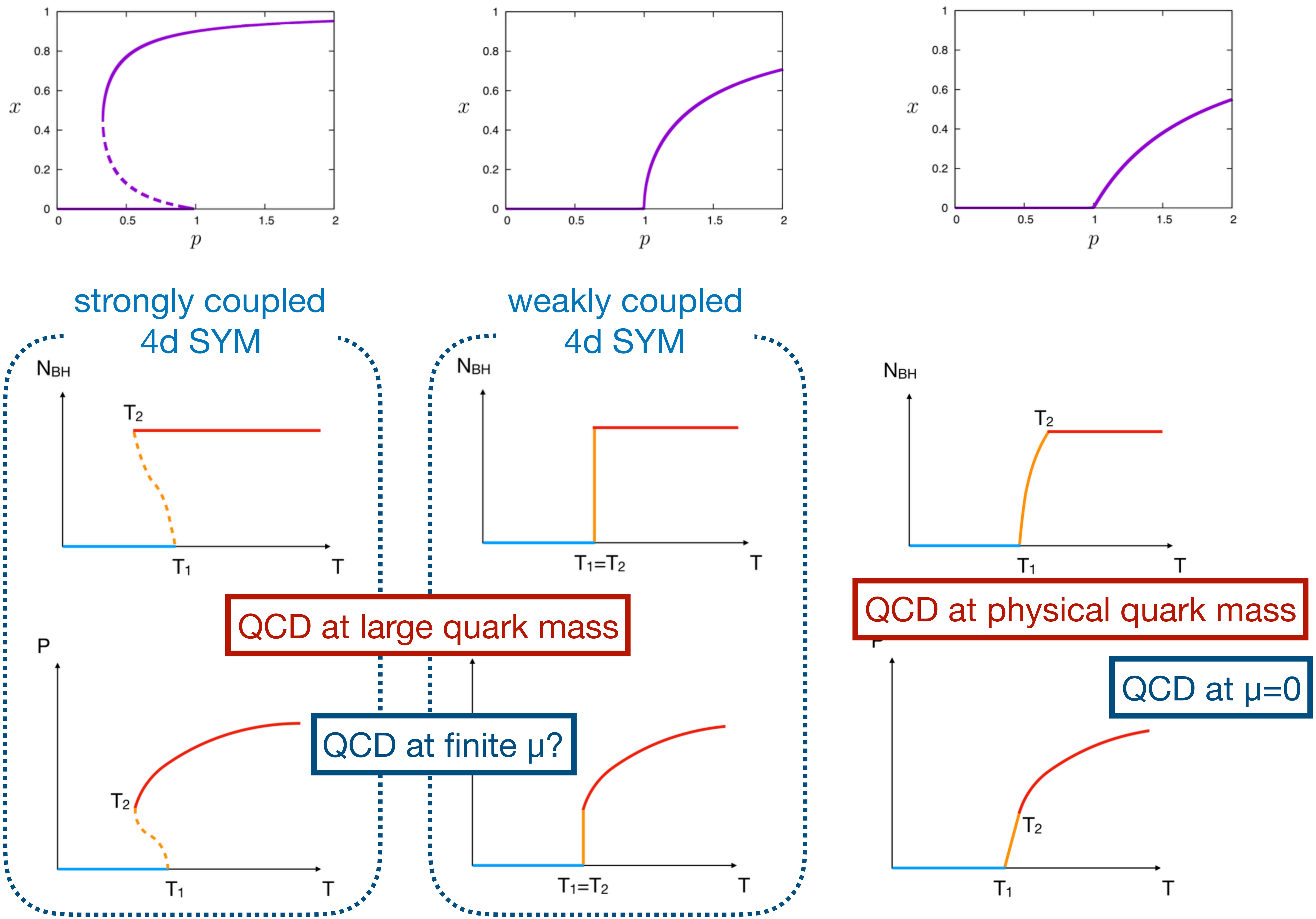
x
p
0
0.5
1
1.5
2
0.2
0.4
0.6
0.8
strongly coupled
4d SYM
weakly coupled
4d SYM
NBH
T2
T1
T
T1=T2
P



x
p
0
0.2
0.4
0.6
0.8
1
0.5
1.5
2
strongly coupled
4d SYM
weakly coupled
4d SYM
NBH
T2
T1
T
T1=T2
P

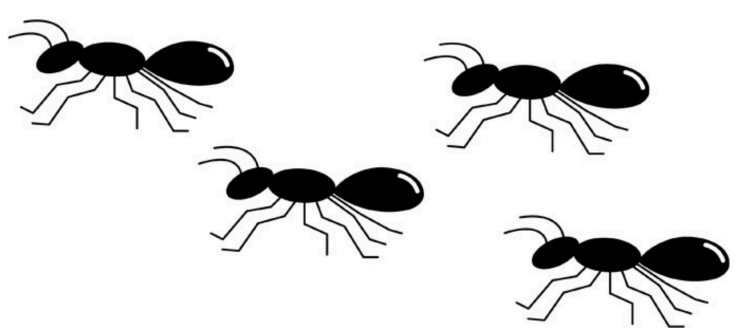


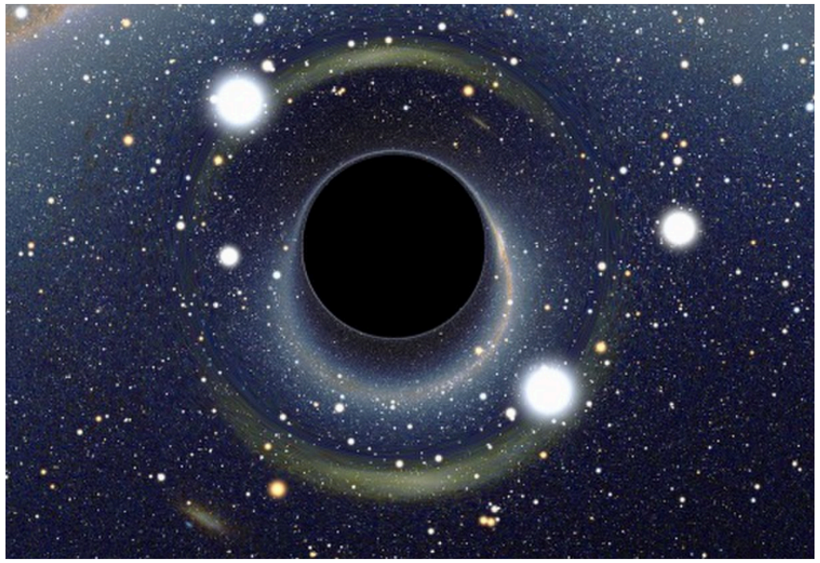
strongly coupled
4d SYM
weakly coupled
4d SYM
QCD at large quark mass
QCD at physical quark mass
x
p
N_BH
P
T
T_1
T_2
T_1=T_2



x
p
strongly coupled
4d SYM
weakly coupled
4d SYM
NBH
T2
T1
T
T1=T2
P
QCD at large quark mass
QCD at physical quark mass
QCD at μ=0
QCD at finite μ?

Testing the partial deconfinement





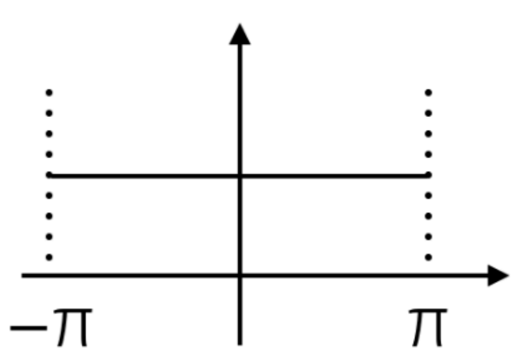
Cotler-MH-Ishiki-Watanabe, in preparation

- ‘Polyakov loop’ is a useful order parameter.

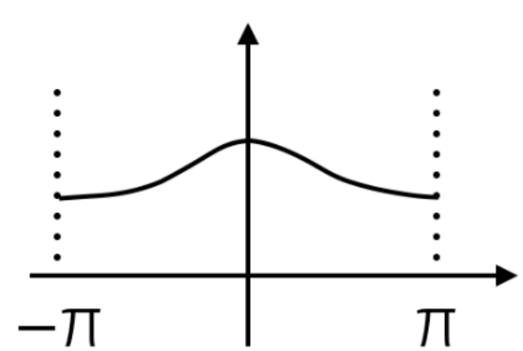
$$P = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

- Phase distribution:

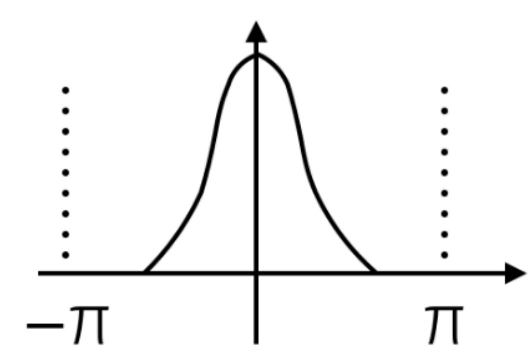
confined phase
P=0



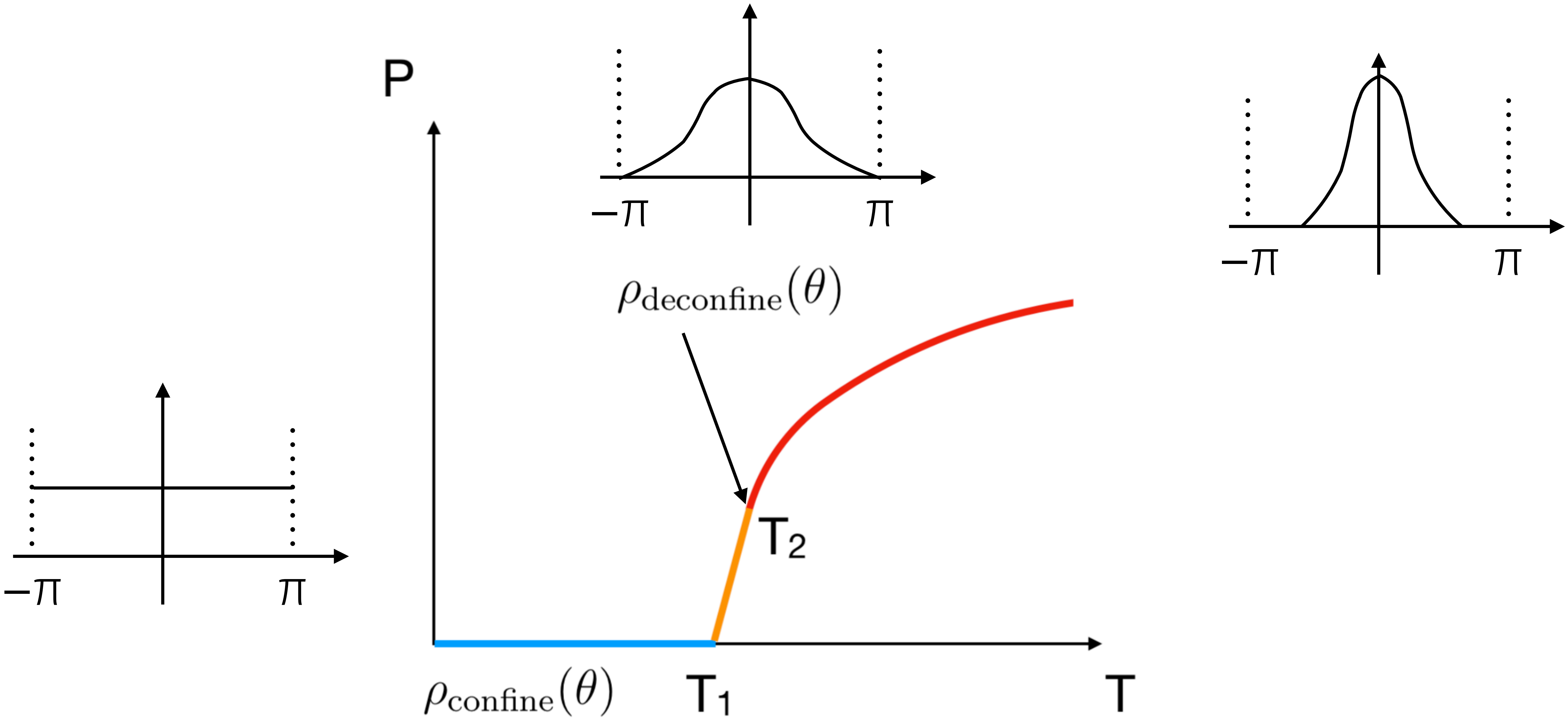
deconfined phase
P ≠ 0



‘partially’ deconfined

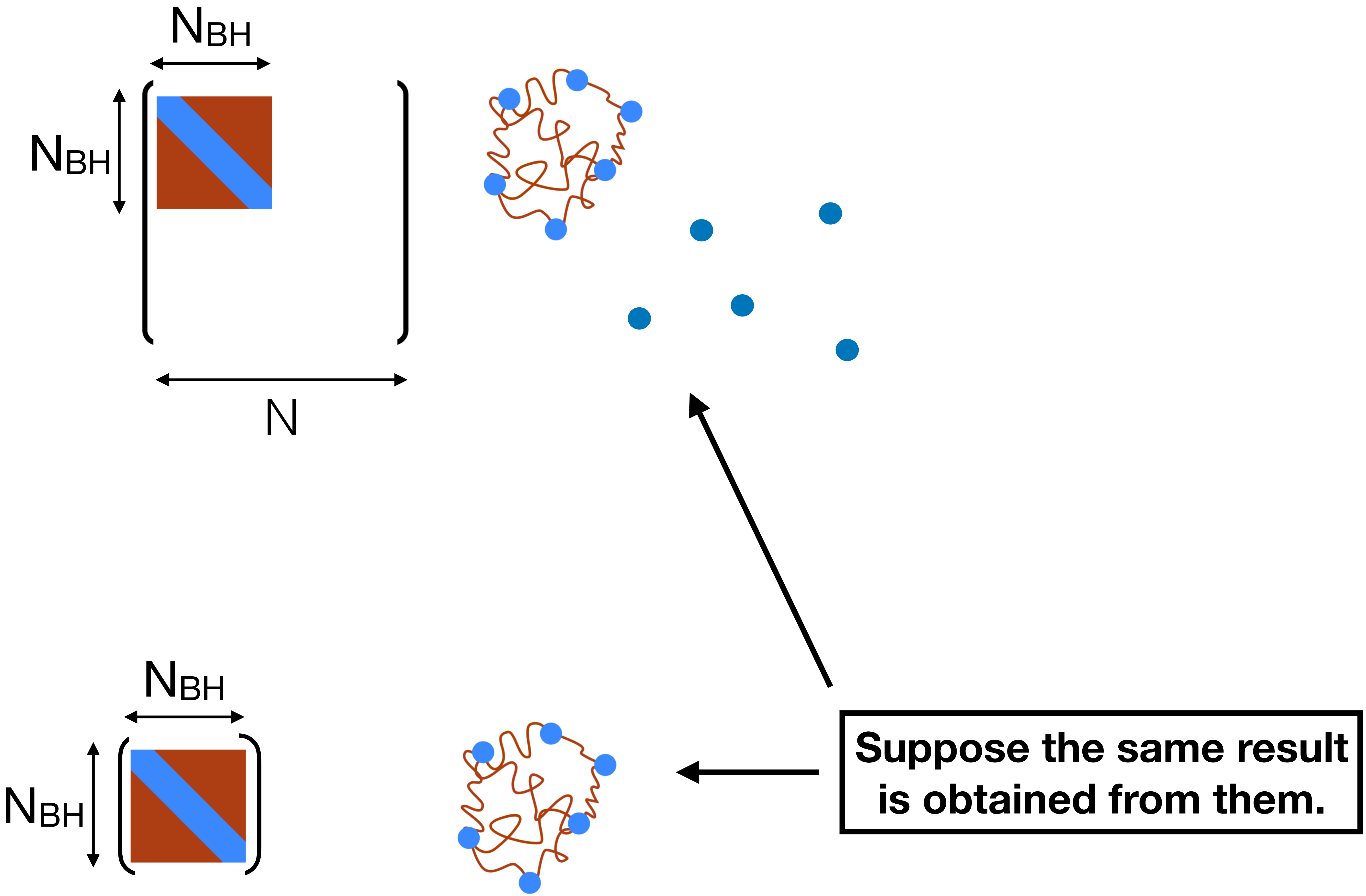


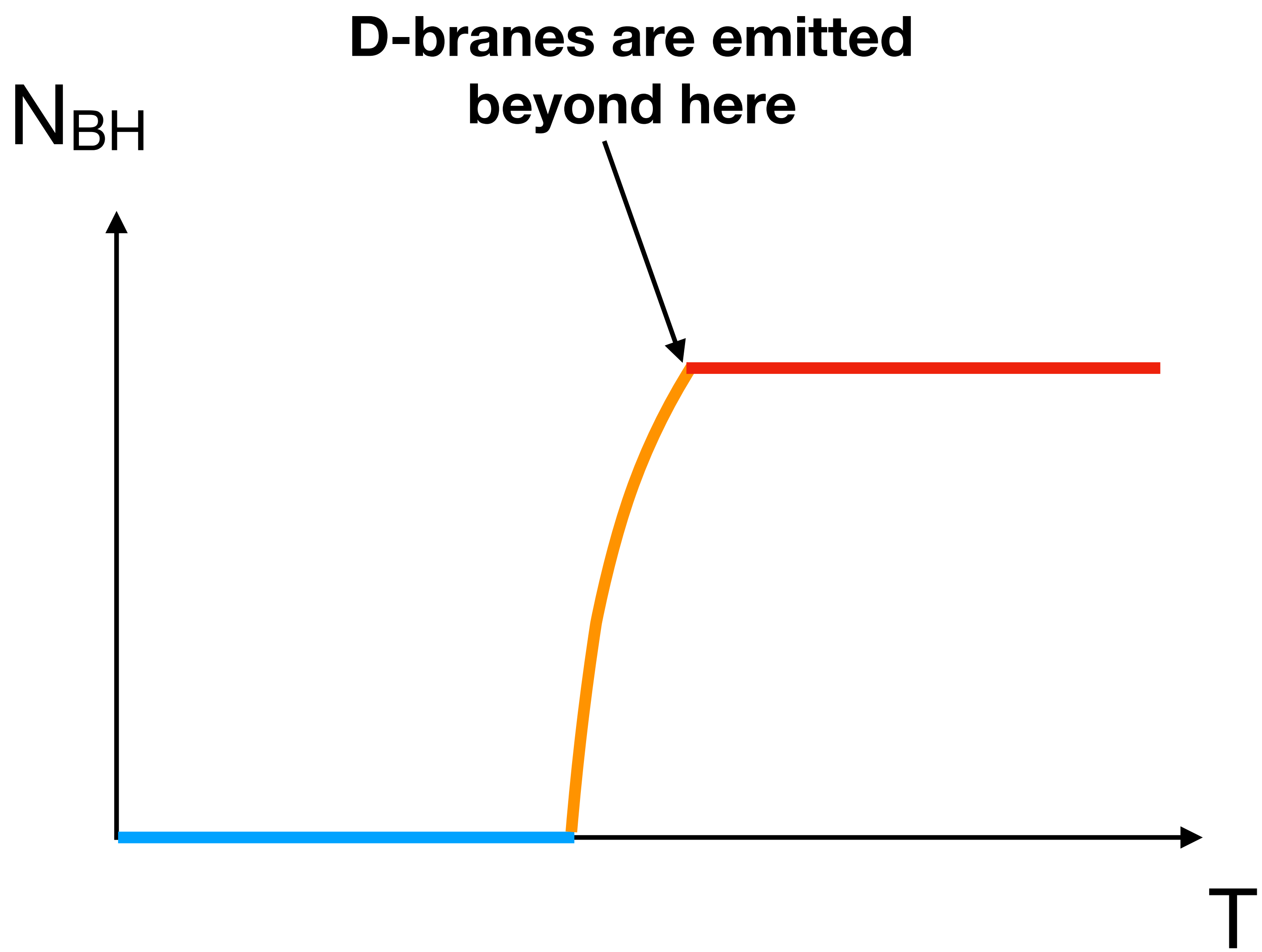
‘completely’ deconfined

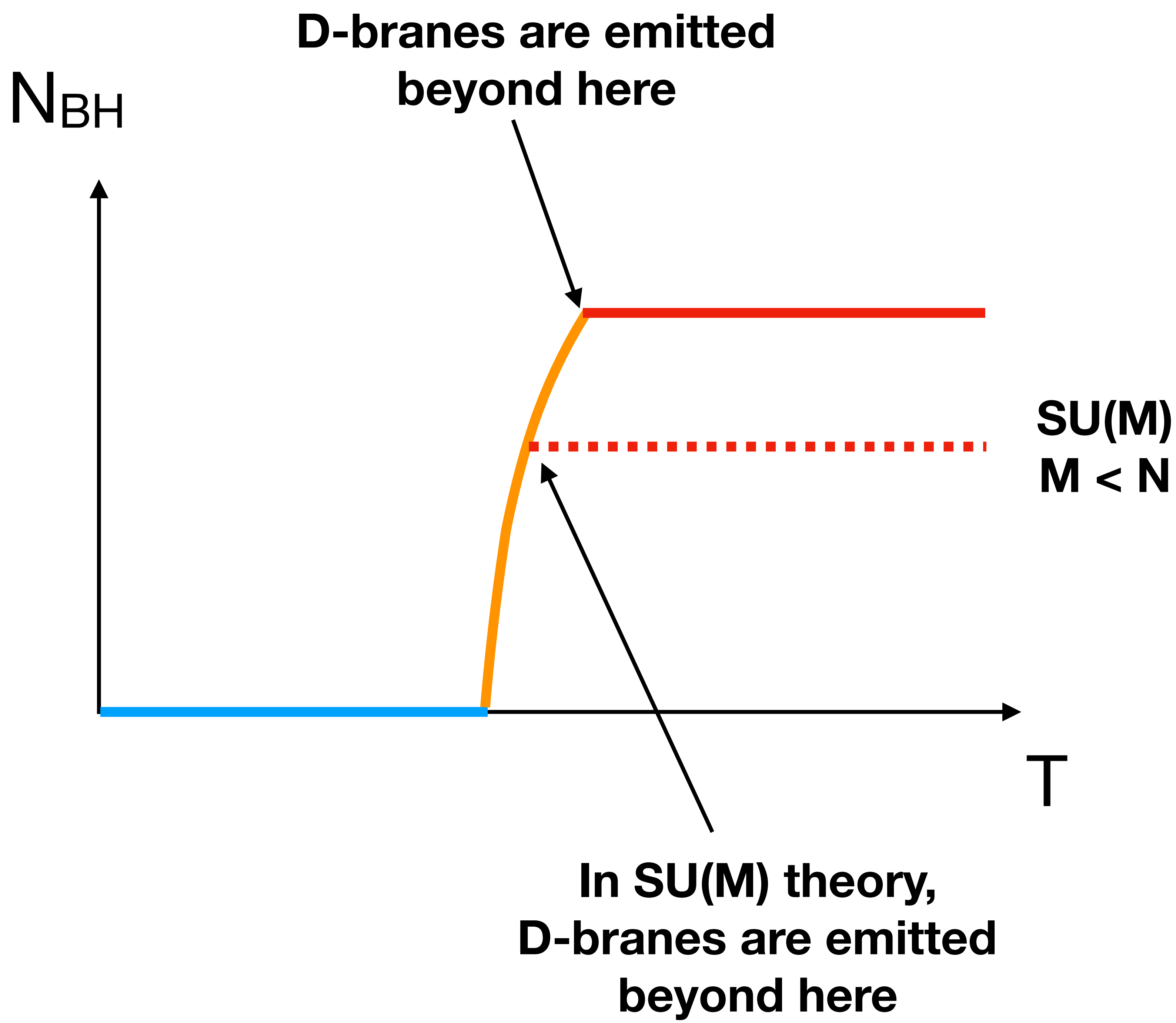


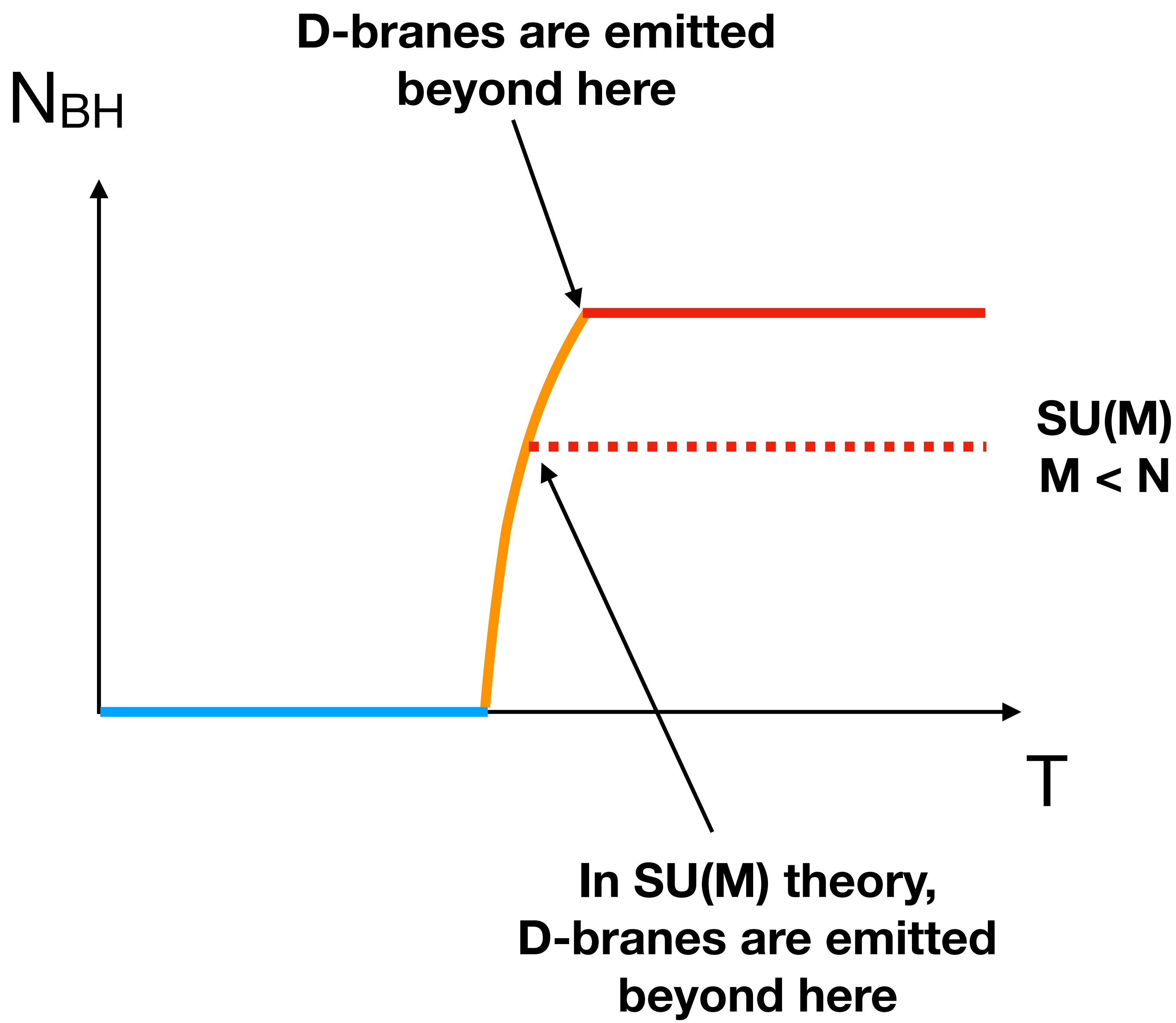
$$\boxed{\rho(\theta) = \frac{N-M}{N}\rho_{\text{confine}}(\theta) + \frac{M}{N}\rho_{\text{deconfine}}(\theta) = \frac{N-M}{N}\cdot\frac{1}{2\pi} + \frac{M}{N}\rho_{\text{deconfine}}(\theta)}$$

It follows from ‘partial deconfinement’ picture.

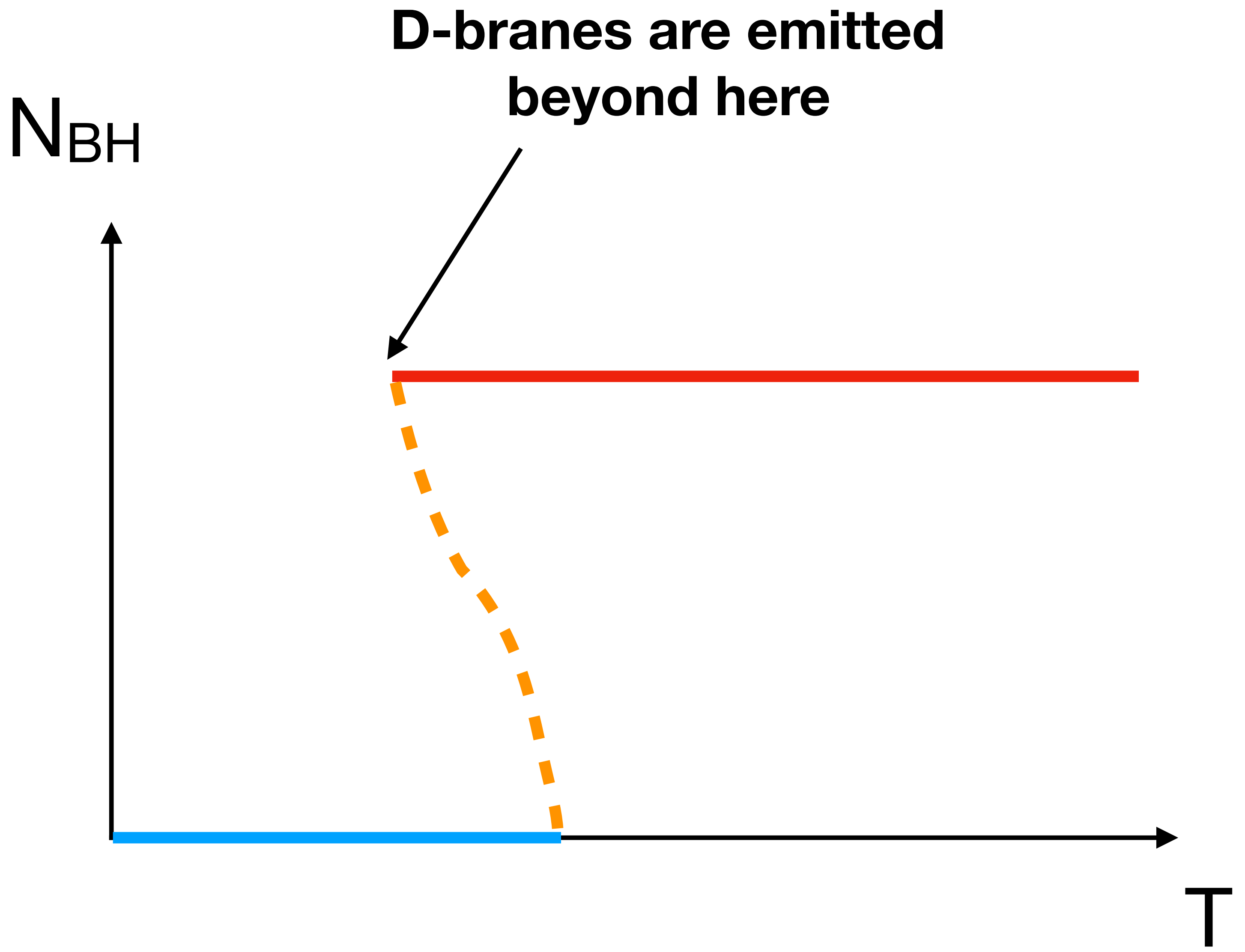

N_{BH}
N_{BH}
N
N_{BH}
N_{BH}
Suppose the same result is obtained from them.


D-branes are emitted
beyond here
N_{BH}
T

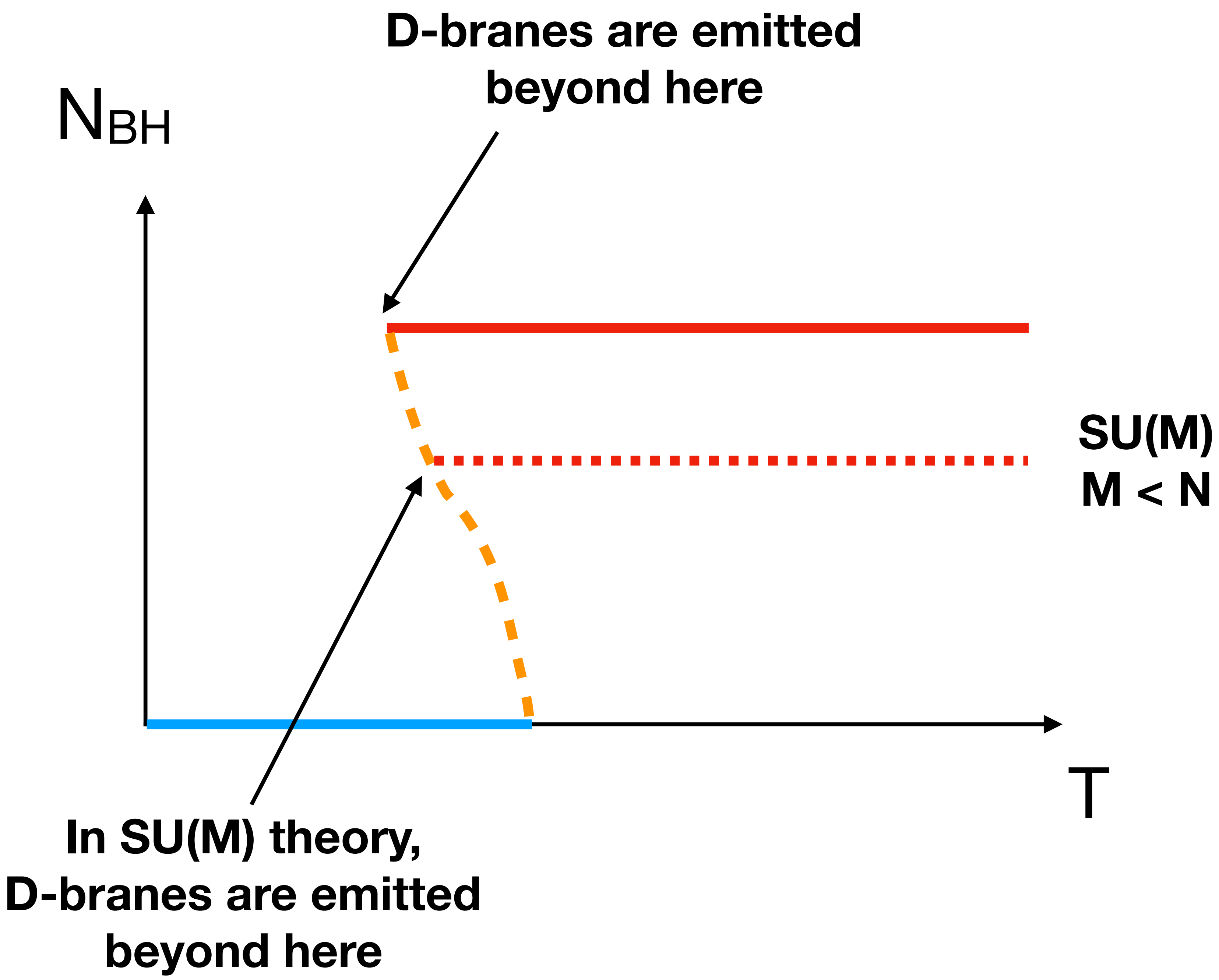

D-branes are emitted
beyond here
N_{BH}
SU(M)
M < N
T
In SU(M) theory,
D-branes are emitted
beyond here

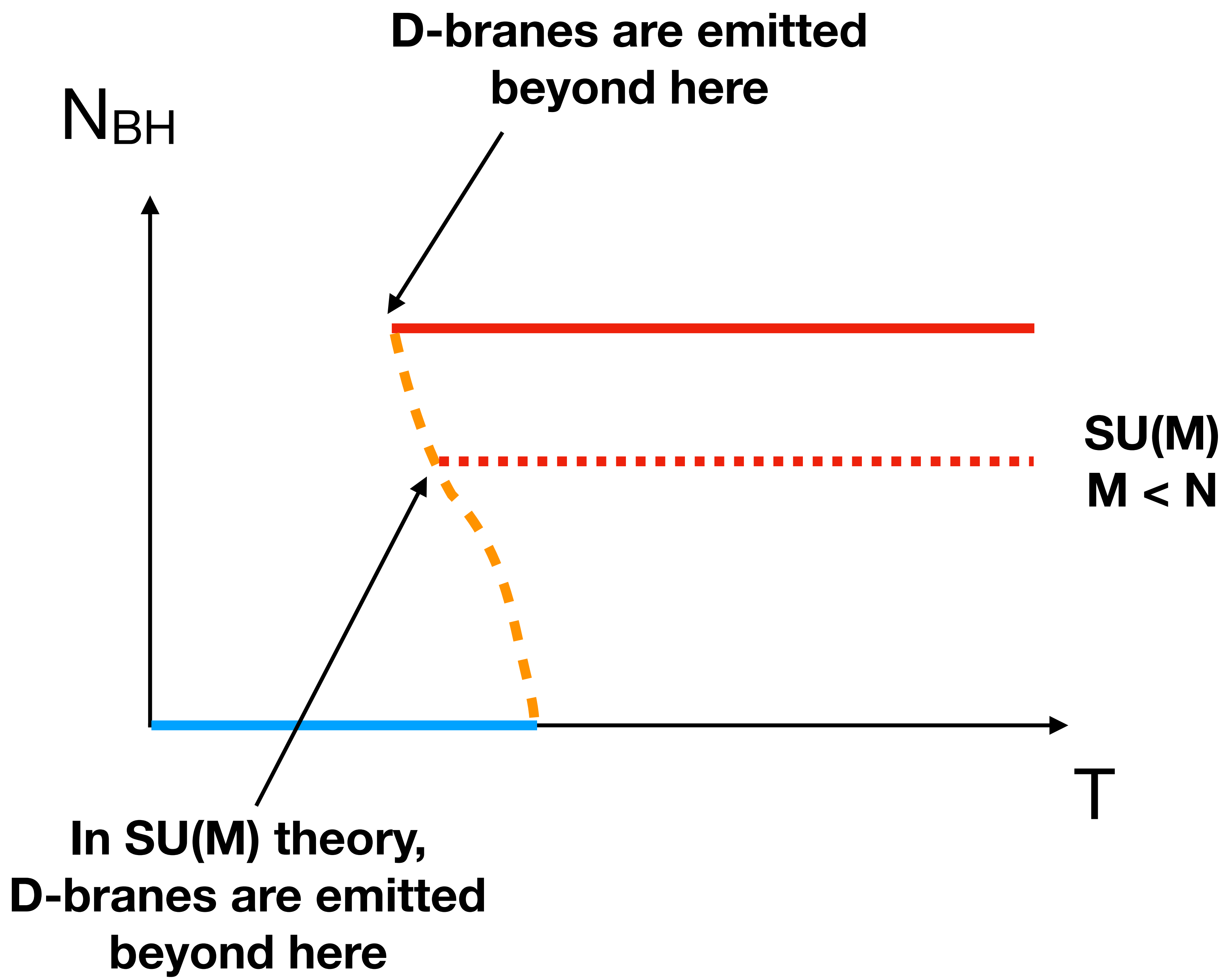


‘Deconfined parts’ behave the same way

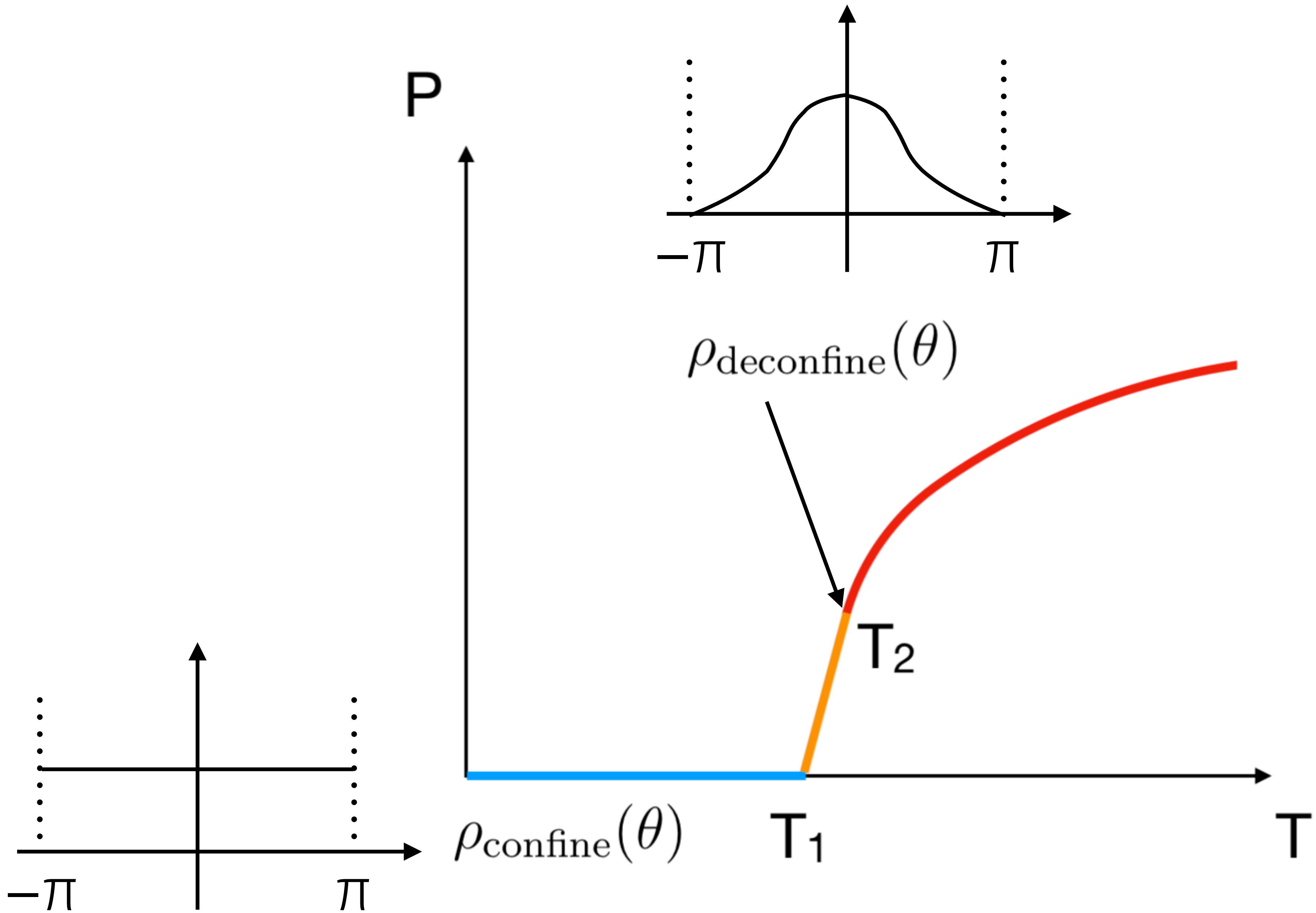


D-branes are emitted
beyond here
N_{BH}
T


D-branes are emitted
beyond here
N BH
SU(M)
M < N
T
In SU(M) theory,
D-branes are emitted
beyond here

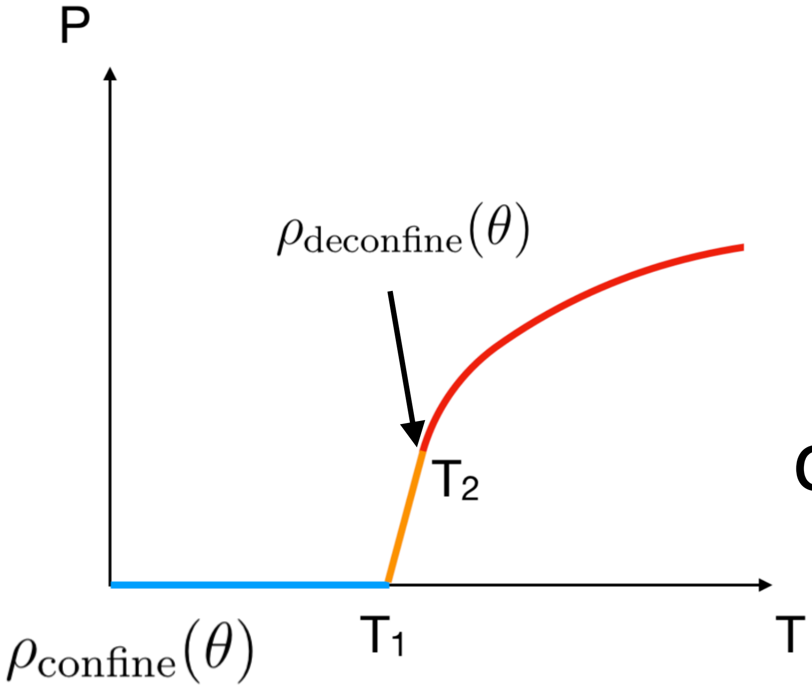


‘Deconfined parts’ behave the same way



$$\boxed{\rho(\theta) = \frac{N-M}{N}\rho_{\text{confine}}(\theta) + \frac{M}{N}\rho_{\text{deconfine}}(\theta) = \frac{N-M}{N}\cdot\frac{1}{2\pi} + \frac{M}{N}\rho_{\text{deconfine}}(\theta)}$$

Does it actually hold?



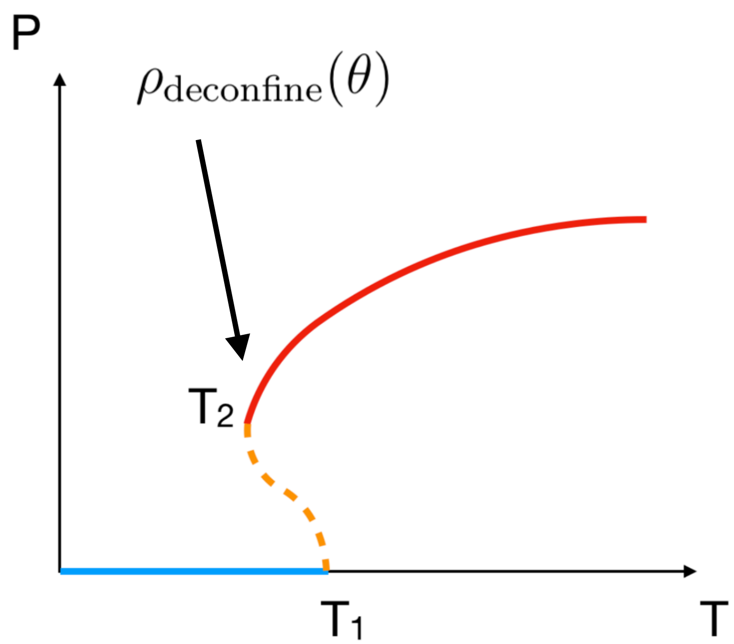
Gross-Witten-Wadia transition separates completely and partially deconfined phases.

$$\rho(\theta)=\begin{cases}\frac{1}{2\pi} & (T\le T_1)\\ \frac{1}{2\pi}\left(1+\frac{2}{\kappa}\cos\theta\right) & (T_1<T<T_2)\\ \frac{2}{\pi\kappa}\cos\frac{\theta}{2}\sqrt{\frac{\kappa}{2}-\sin^2\frac{\theta}{2}} & \left(T\ge T_2,\ |\theta|<2\arcsin\sqrt{\kappa/2}\right)\end{cases}$$

$$\boxed{\rho(\theta)=\frac{N-M}{N}\rho_{\rm confine}(\theta)+\frac{M}{N}\rho_{\rm deconfine}(\theta)=\frac{N-M}{N}\cdot\frac{1}{2\pi}+\frac{M}{N}\rho_{\rm deconfine}(\theta)}$$

It does hold in various examples.

$$\frac{M}{N}=\frac{2}{\kappa}$$



not tested yet

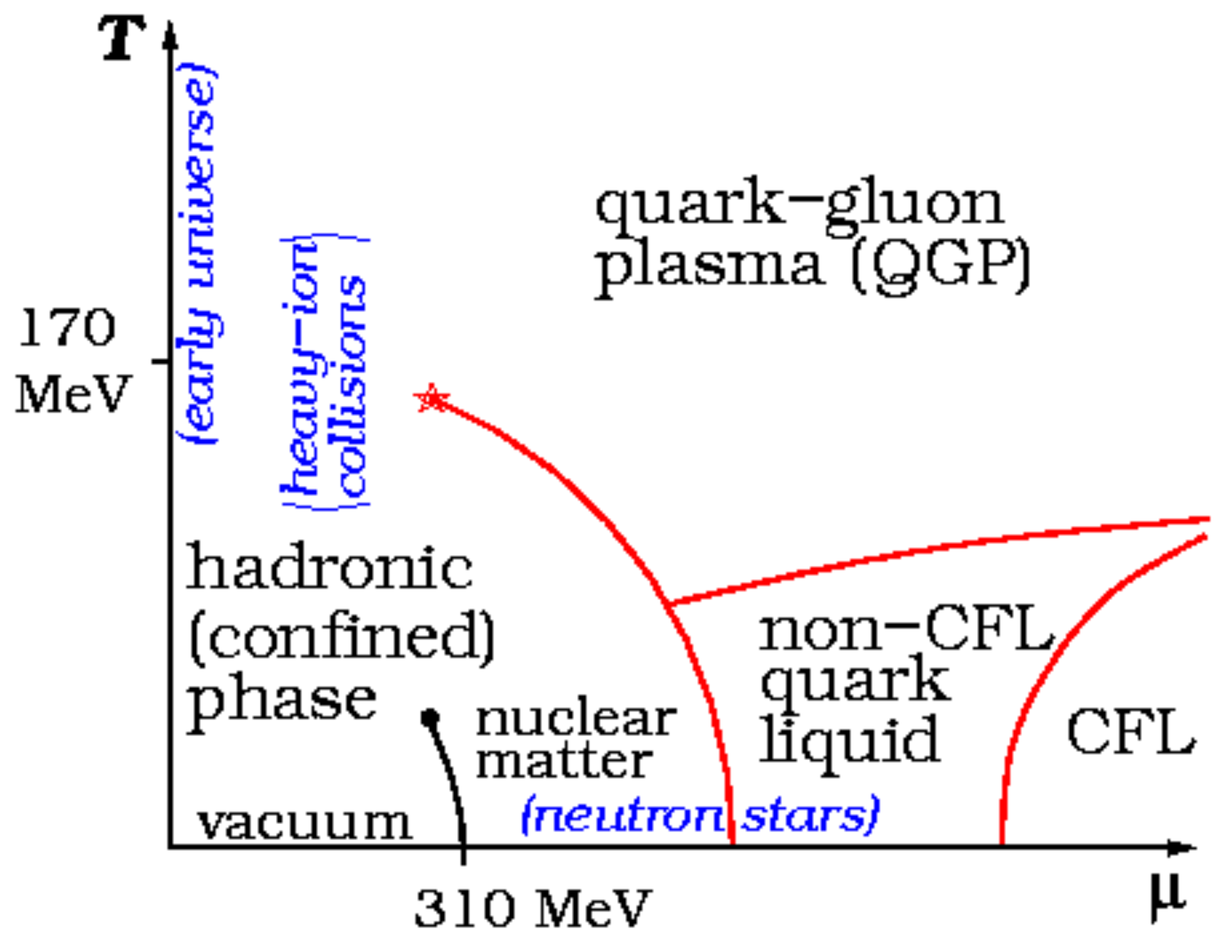
$T_2 < T_1$

$$\rho(\theta) = \begin{cases} \dfrac{1}{2\pi} & (T \le T_1) \\ \boxed{\dfrac{1}{2\pi}\left(1 + \dfrac{2}{\kappa}\cos\theta\right)} & \boxed{(T_1 < T < T_2)} \\ \boxed{\dfrac{2}{\pi\kappa}\cos\dfrac{\theta}{2}\sqrt{\dfrac{\kappa}{2} - \sin^2\dfrac{\theta}{2}}} & (T \ge T_2, |\theta| < 2\arcsin\sqrt{\kappa/2}) \end{cases}$$

It does hold in various examples.

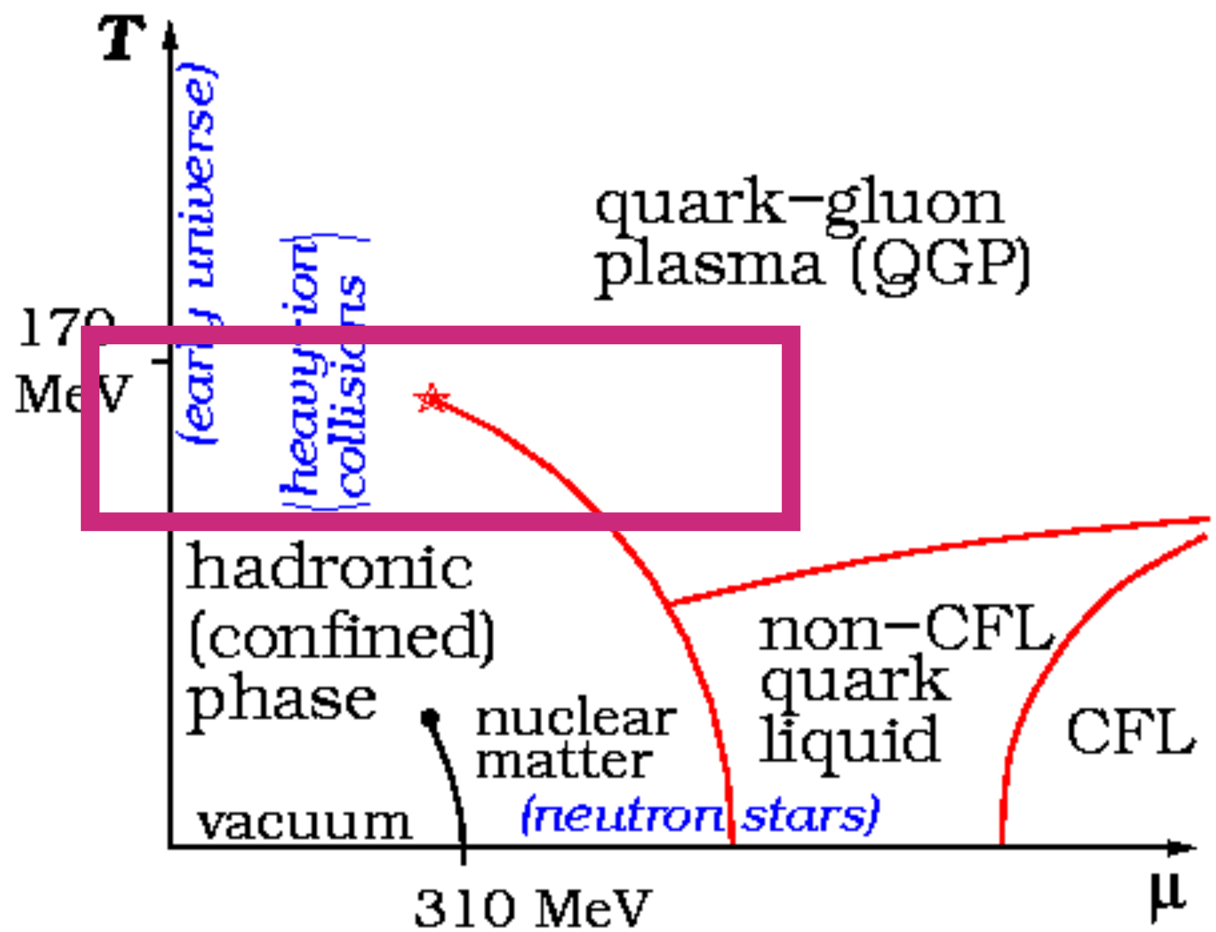
Finite density QCD
for
Hawking Evaporation?

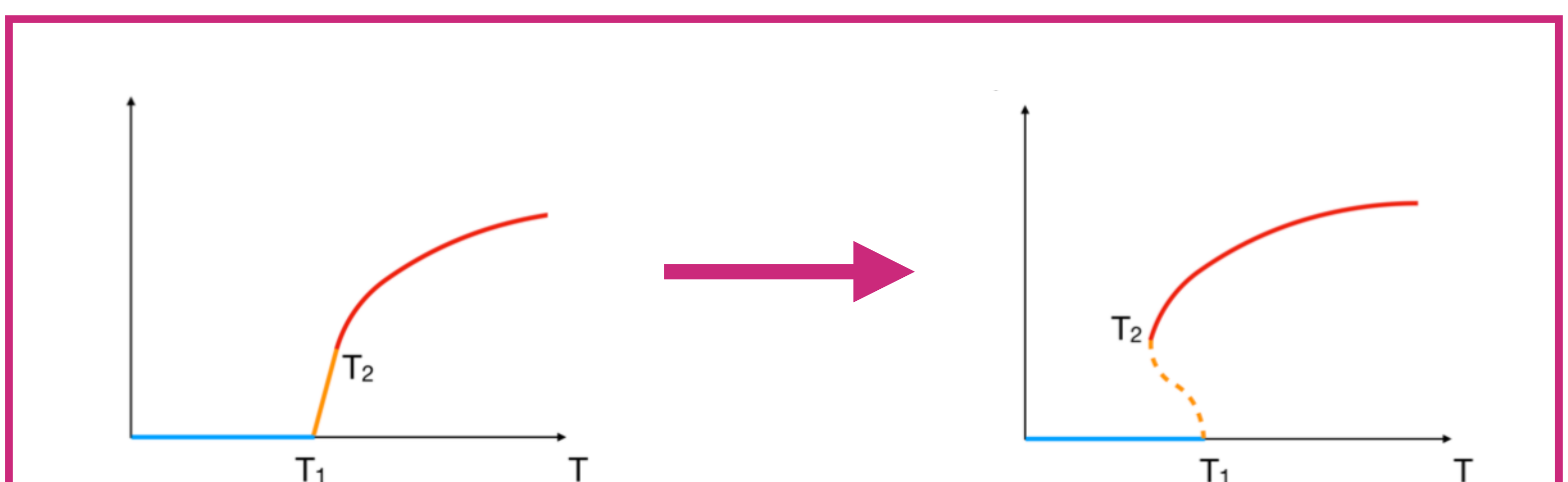
Conjectured QCD phase diagram



(from Wikipedia)

Conjectured QCD phase diagram



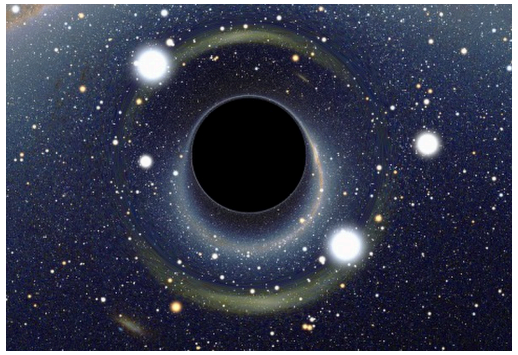


(from Wikipedia)

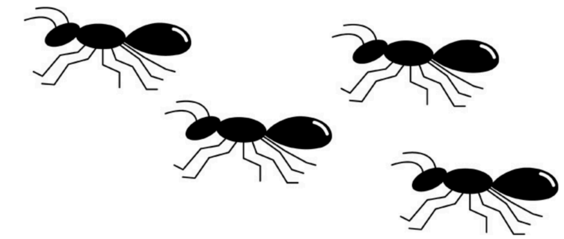
- ‘Evaporating black hole’ should be there.

disclaimer: ‘Gravity dual’ can be very stringy.

- What would be the experimental signal?
- ‘Applied holography’ should be a good tool.



Conclusion



- Ants are smart. They know many things about black hole.

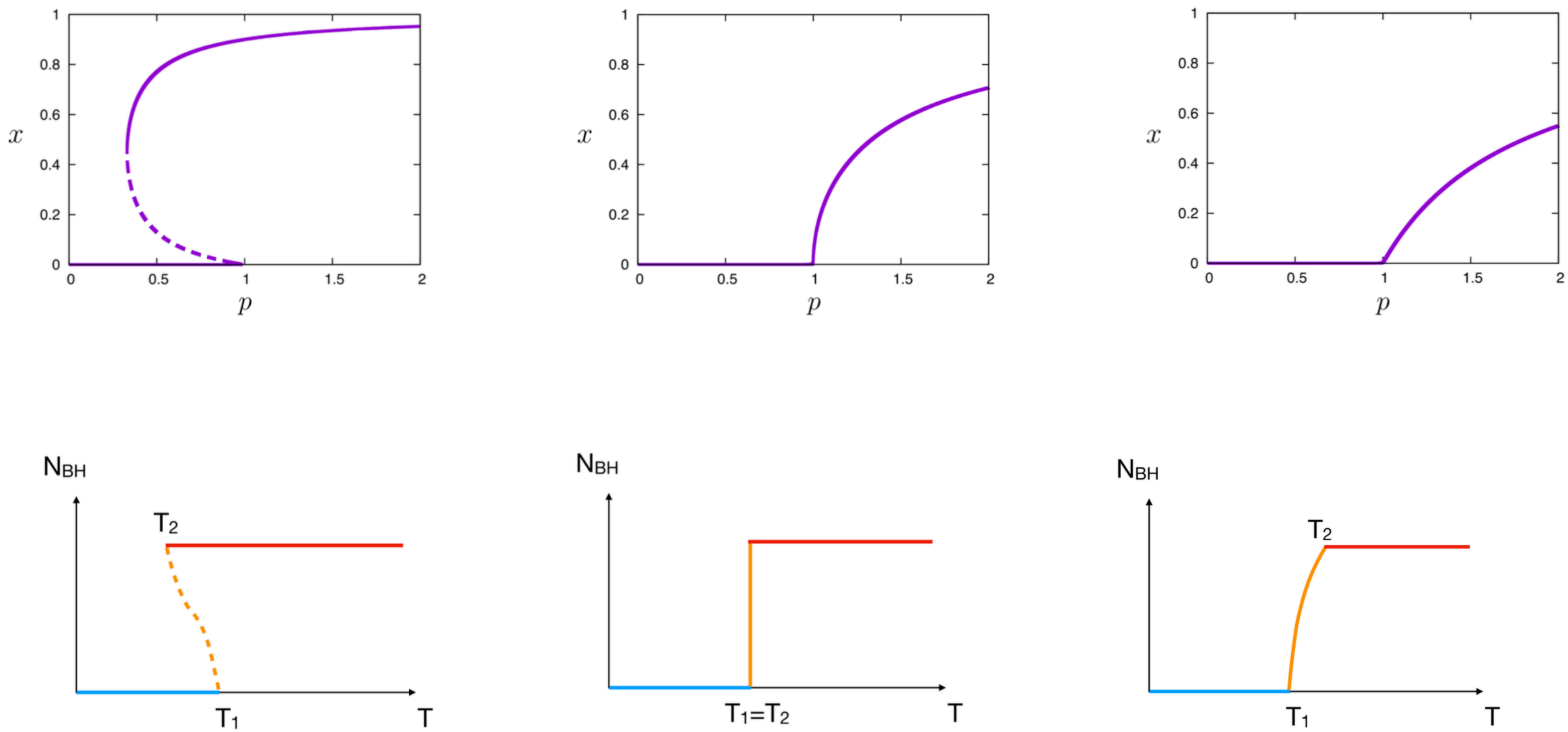
Lesson #2: Take "coincidences" seriously.

- 'Partial deconfinement' and 'Schwarzschild Black Hole' are rather generic in gauge theories.

- 'Hawking evaporation' in the heavy ion collision?

- It is important to study gauge theory, in order to understand quantum gravity.

- Are we smarter than ants?



x
p
0
0.2
0.4
0.6
0.8
1
0.5
1.5
2
N_BH
T
T_1
T_2
T_1=T_2

$x = N_{trail}/N$

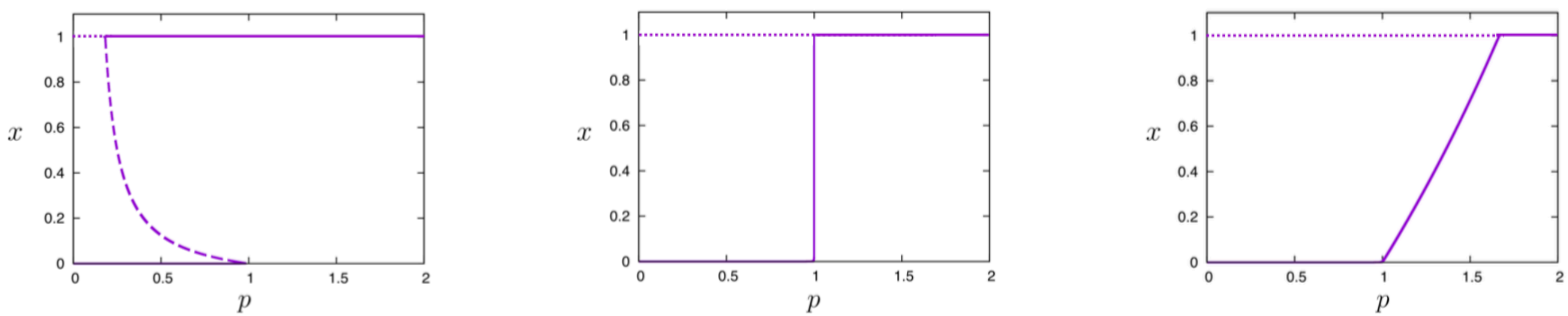
$$\frac{dx}{dt} = (\tilde{\alpha} + px)(1 - x) - \frac{\tilde{s}x}{\tilde{s} + x} \cdot \underline{(1 - x^2)} \ = \mathbf{0}$$

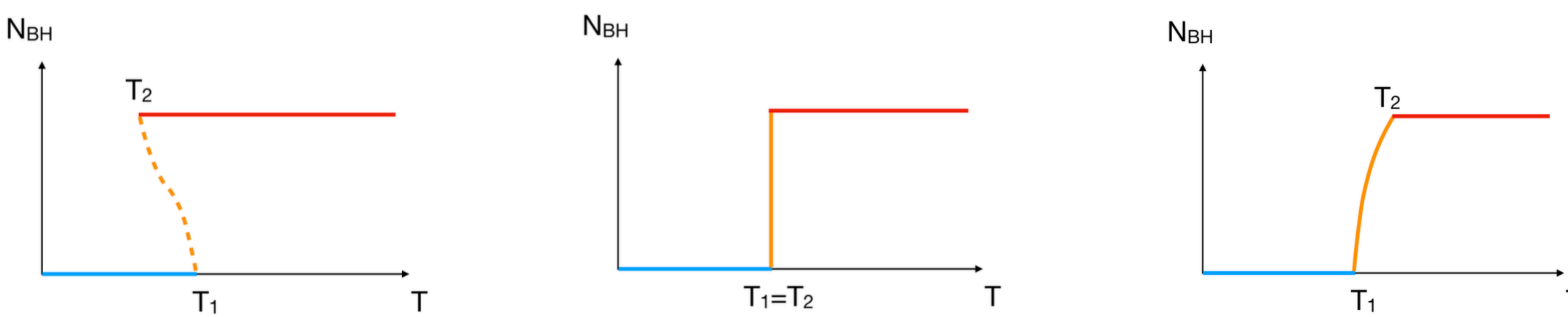
don’t want to be a lone ant

$x = N_{trail}/N$

$$\frac{dx}{dt} = (\tilde{\alpha} + px)(1-x) - \frac{\tilde{s}x}{\tilde{s}+x} \cdot \underline{(1-x^2)} \;\; \mathbf{= 0}$$

don’t want to be a lone ant



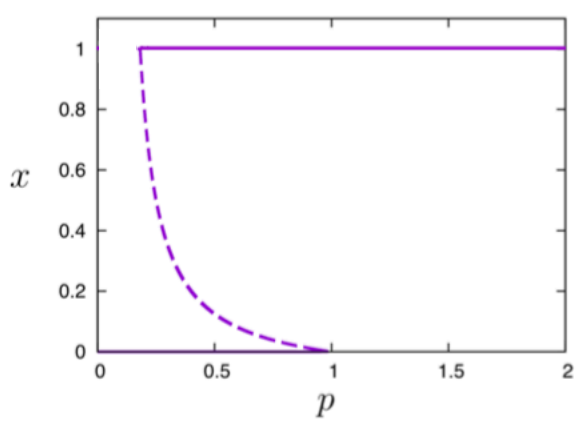


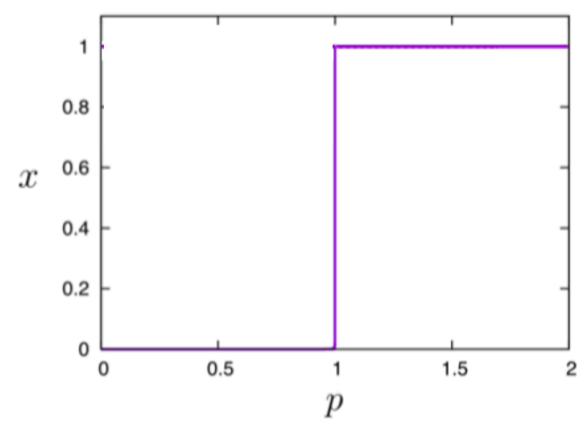
+ε

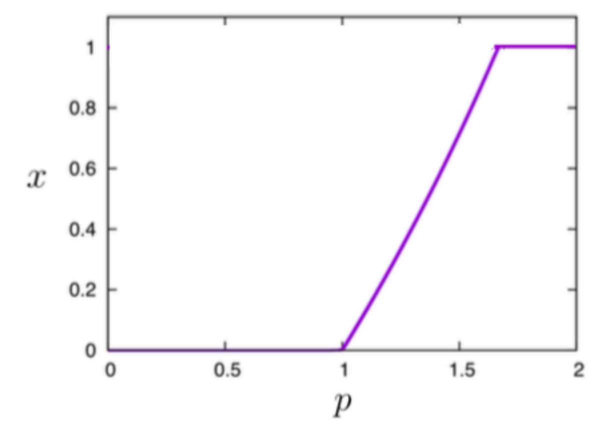
$x = N_{trail}/N$

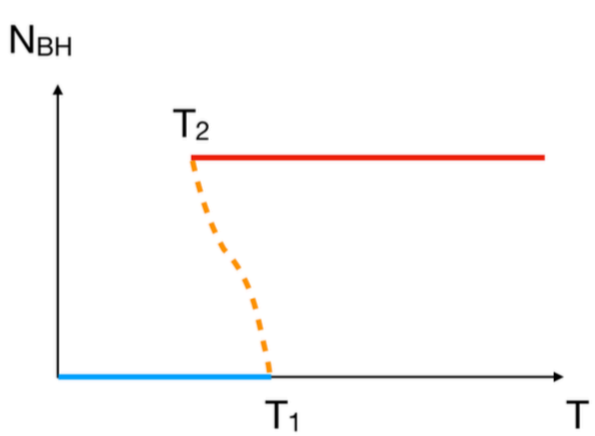
$$\frac{dx}{dt} = (\tilde{\alpha} + px)(1-x) - \frac{\tilde{s}x}{\tilde{s}+x} \cdot \underline{(1-x^2)} = 0$$

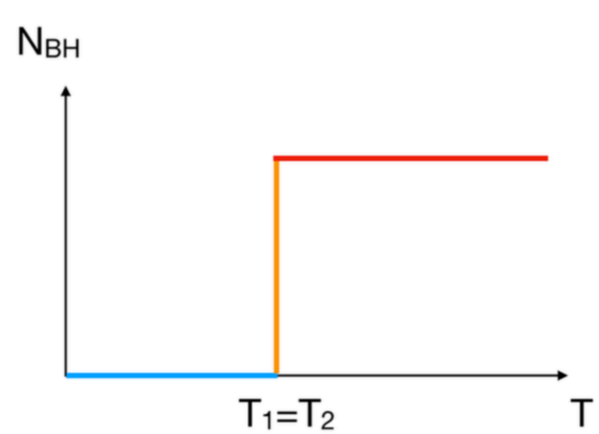
don't want to be a lone ant

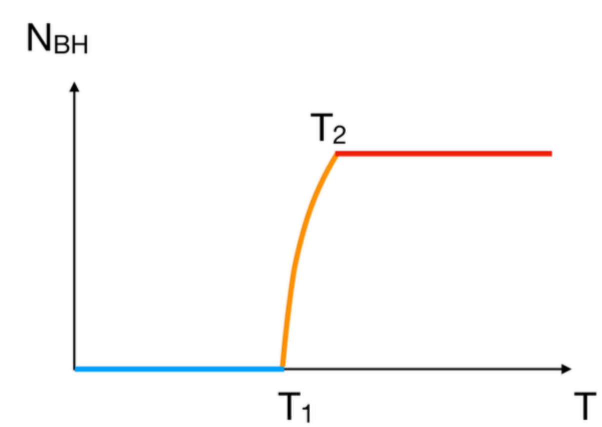












Backup Slides

10d Schwarzschild from 4d SYM

via

Partial Deconfinement

M.H., Maltz, 2016

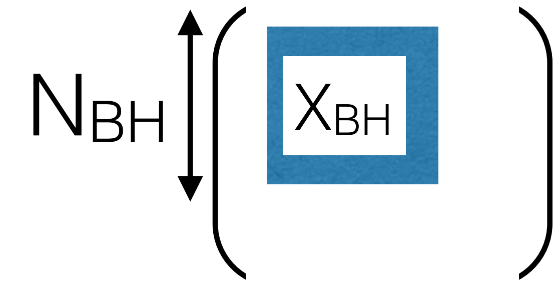
Heuristic Gauge Theory 'Derivation' (1)

- Take radius of S^3 to be 1.
- At strong coupling, the interaction term $(N/\lambda)*\mathrm{Tr}[X_I,X_J]^2$ is dominant.

$\lambda=g_{YM}^2N$

- Eigenvalues of $Y = \lambda^{-1/4}X$ are $O(1)$ because the interaction is simply $N*\mathrm{Tr}[Y_I,Y_J]^2$.
- Hence eigenvalues of X are $O(\lambda^{1/4})$.

Heuristic Gauge Theory 'Derivation' (2)



- When bunch size shrinks to $N_{BH}<N$, 't Hooft coupling effectively becomes $\lambda_{BH}=g_{YM}^2 N_{BH}$

$\lambda=g_{YM}^2 N$

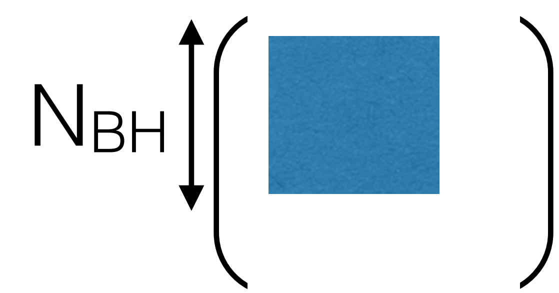
- Hence eigenvalues of X_{BH} are $O(\lambda_{BH}^{1/4}) = O(g_{YM}^{1/2} N_{BH}^{1/4})$.

- $E_{BH} \sim N_{BH}^2 (N_{BH}/N)^{-1/4}$, $S_{BH} \sim N_{BH}^2$

- $T_{BH} \sim (N_{BH}/N)^{-1/4}$

Heuristic Gauge Theory 'Derivation' (3)

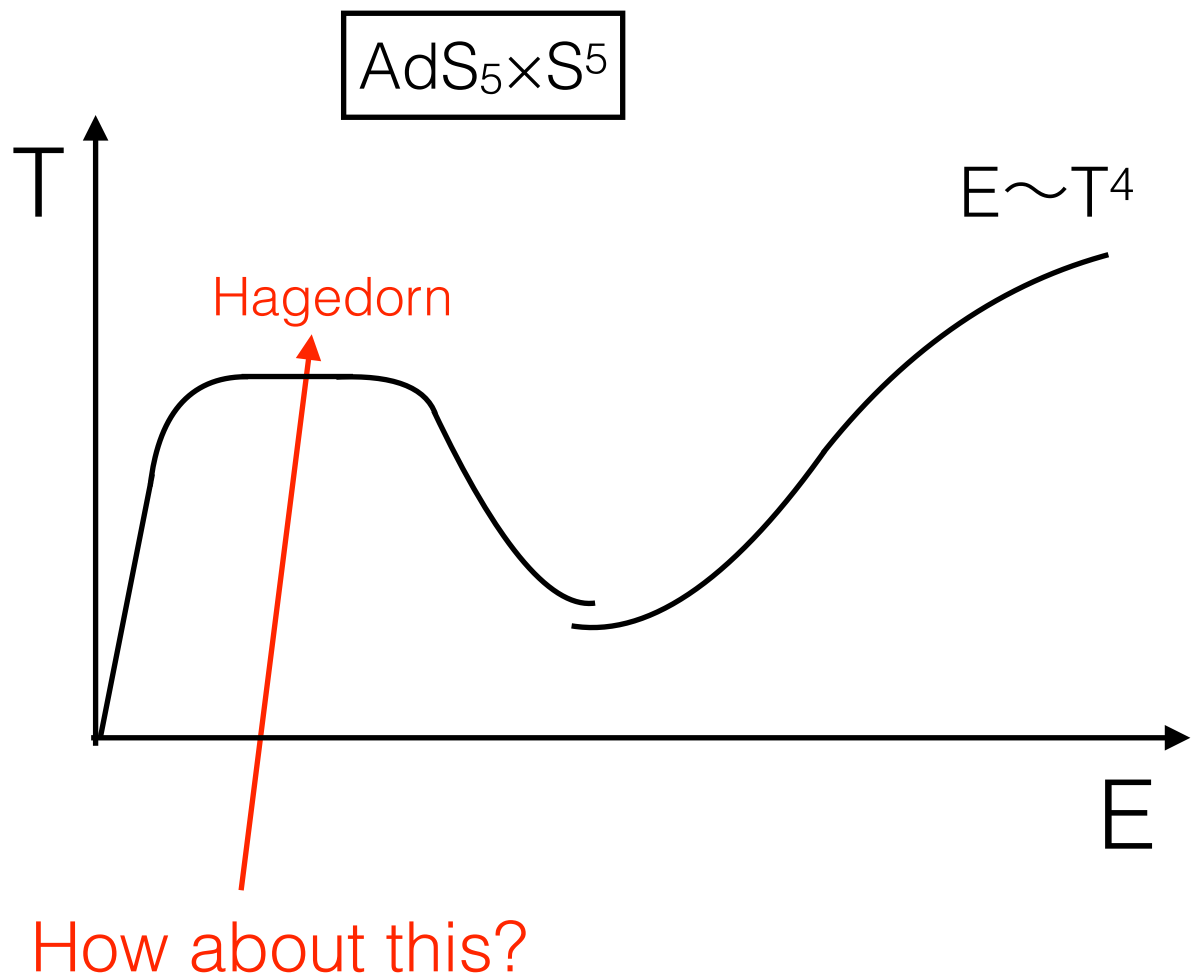
- $E_{BH} \sim N_{BH}^2 (N_{BH}/N)^{-1/4}$, $S_{BH} \sim N_{BH}^2$
- $T_{BH} \sim (N_{BH}/N)^{-1/4}$

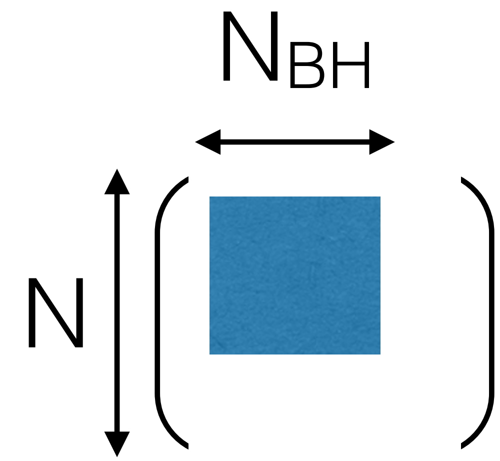


- $E_{BH} \sim N^2 (N_{BH}/N)^{7/4} \sim 1/(G_{N,10} T_{BH}^7)$
- $S_{BH} \sim N^2 (N_{BH}/N)^2 \sim 1/(G_{N,10} T_{BH}^8)$

10d Schwarzschild

- The same logic applied to M-theory region of ABJM gives 11d Schwarzschild, $E \sim 1/G_{N,11} T^8$.





$E < E_{min}$

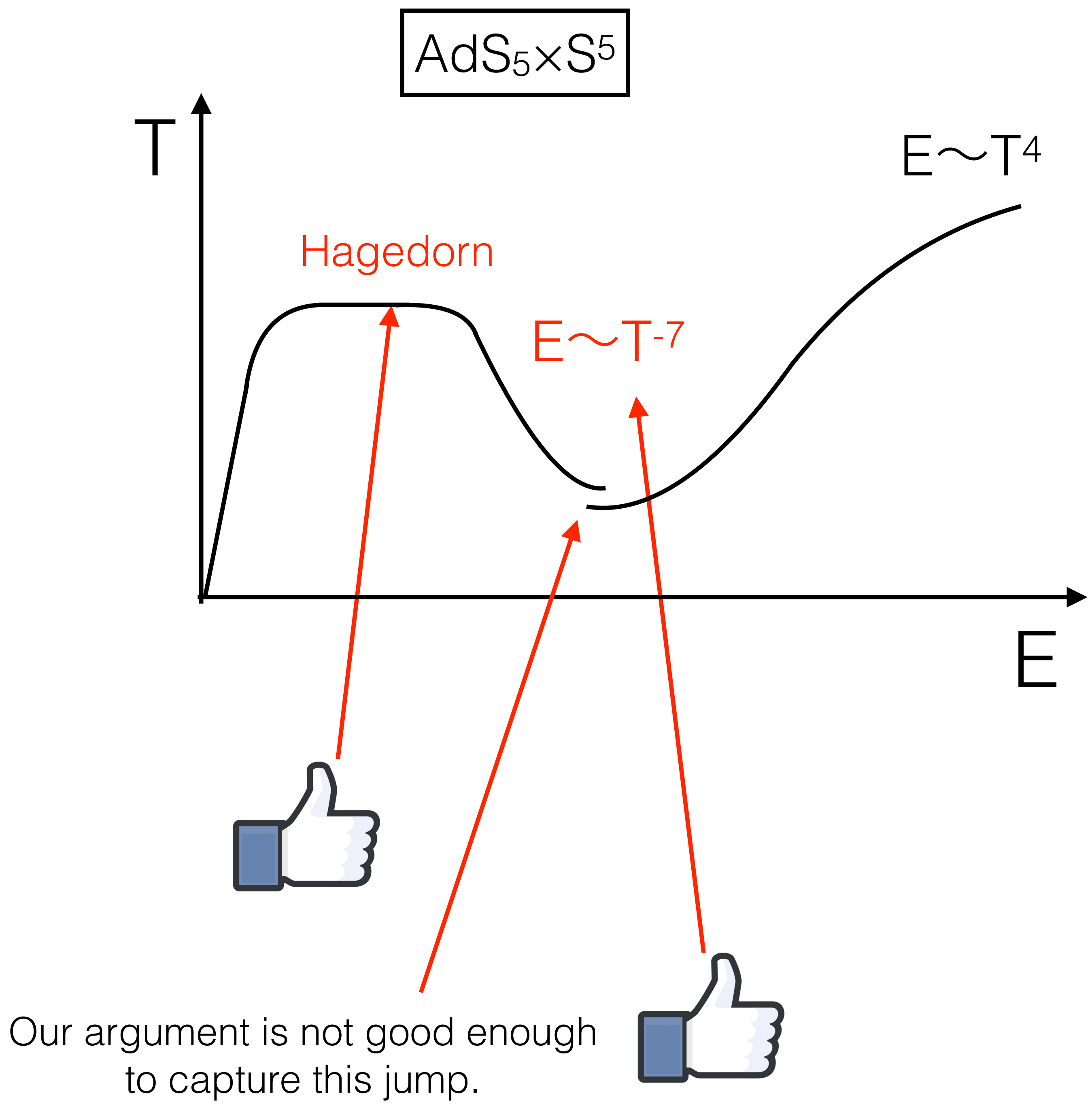
$$T_{BH}=T_{Hagedorn}\sim 1$$

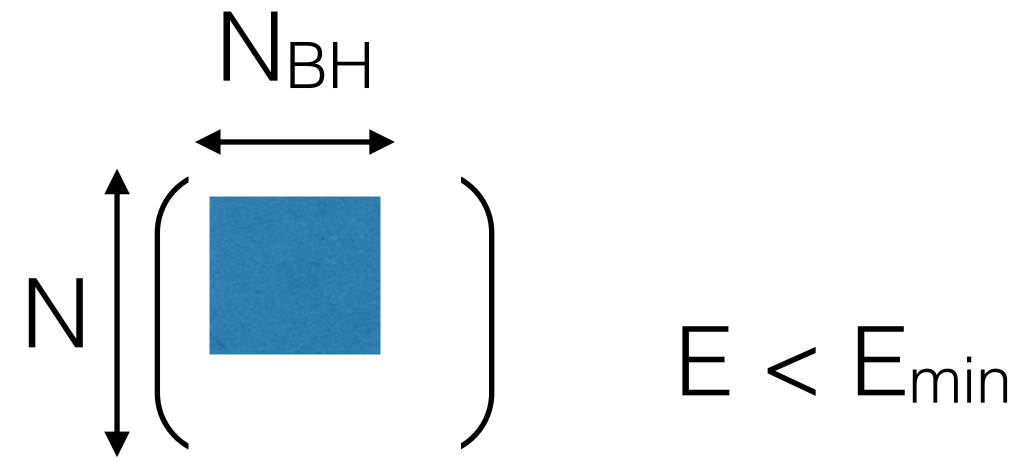
$$E_{BH}\sim S_{min}\sim N_{BH}^2$$

when $g_{YM}^2 N_{BH} << 1$

Just perturbative SYM.

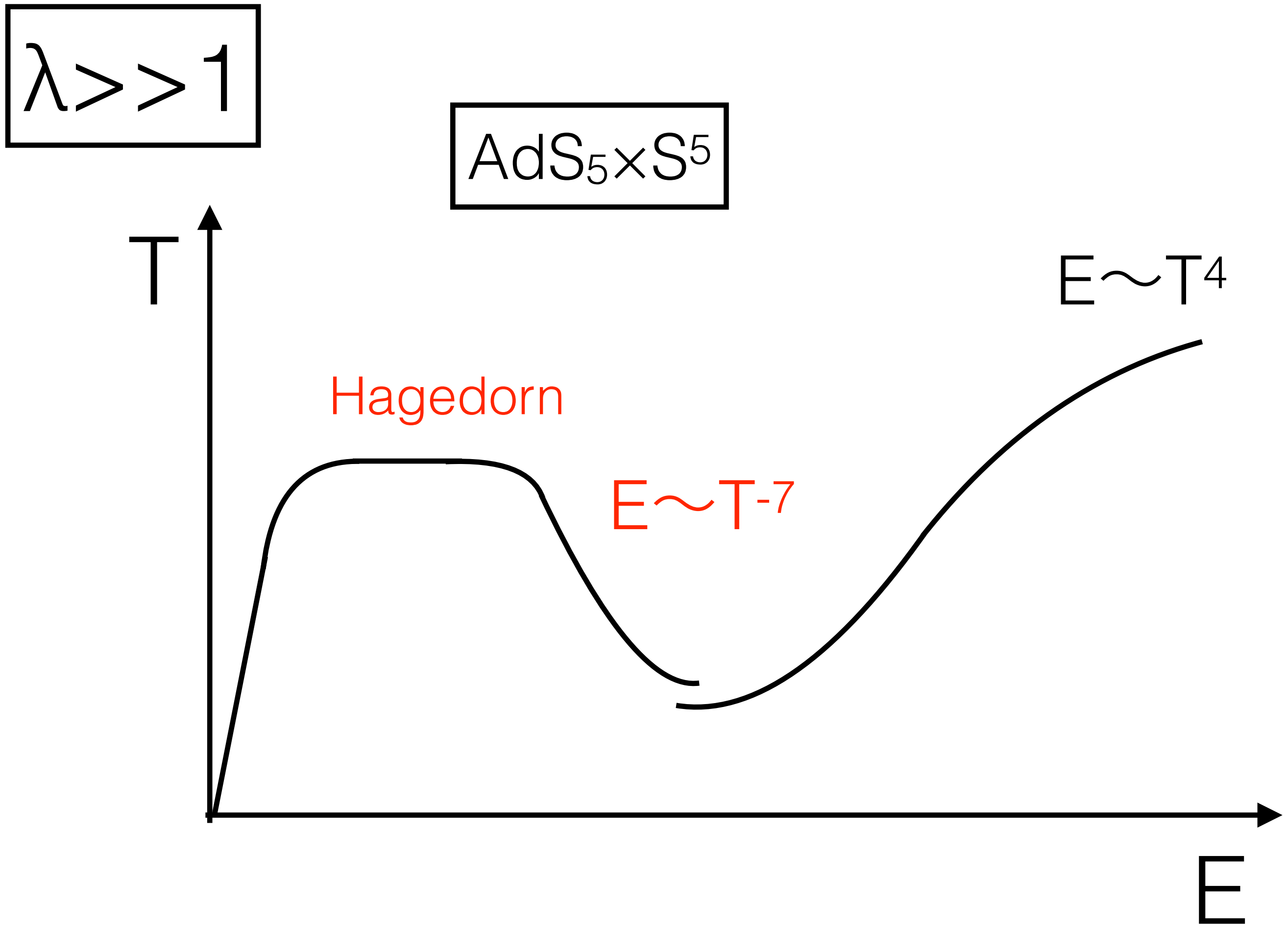
$g_{YM}^2 N_{BH} << 1$

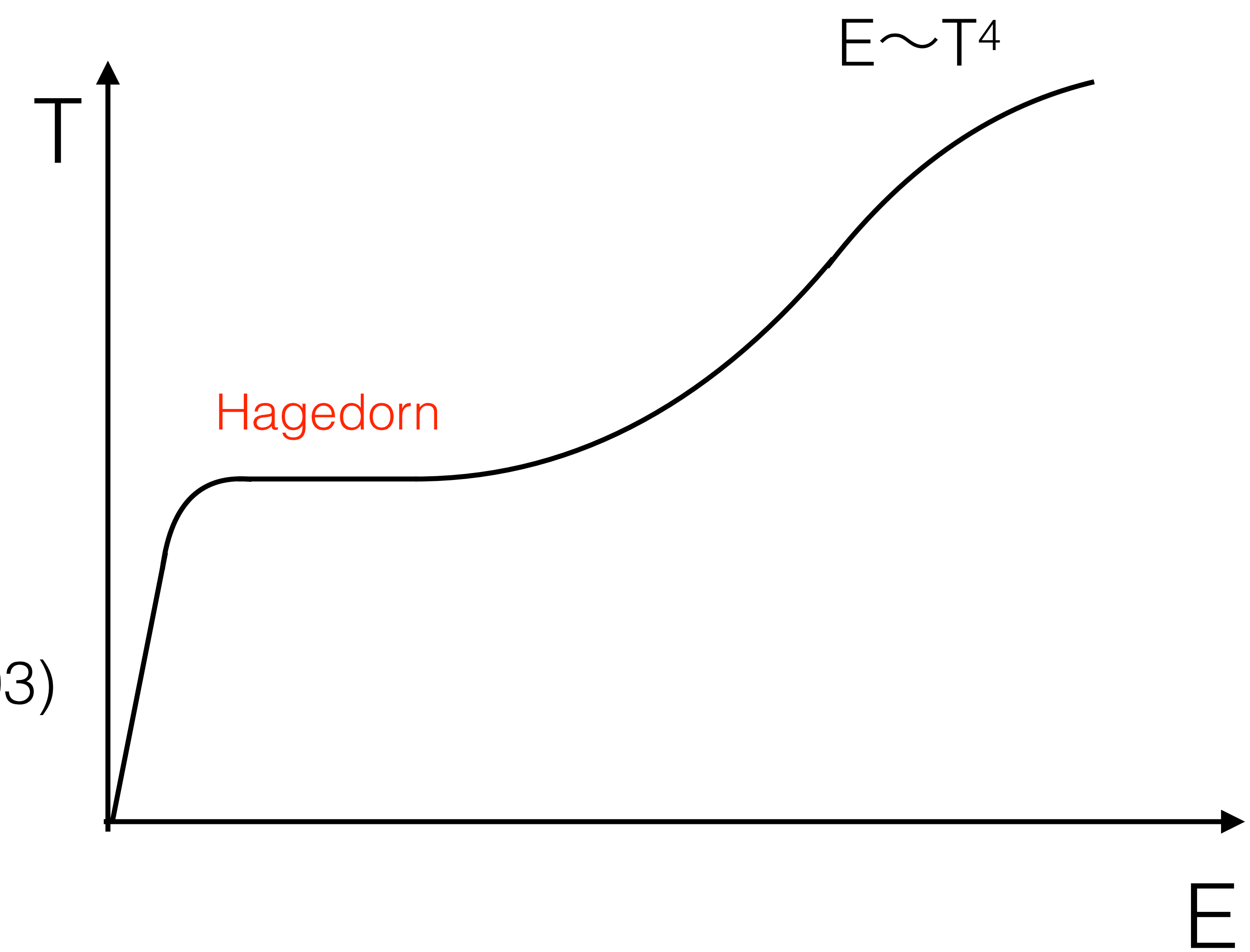




Large BH = ‘Large’ Matrices

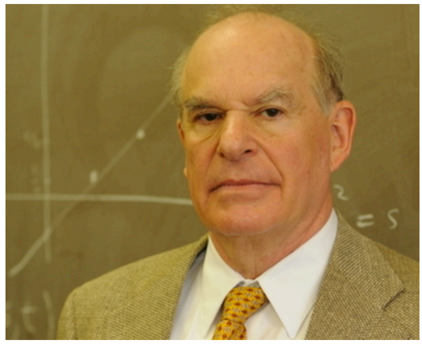
Small BH = ‘Small’ Matrices





$\lambda<<1$

(see e.g. Aharony et al 2003)

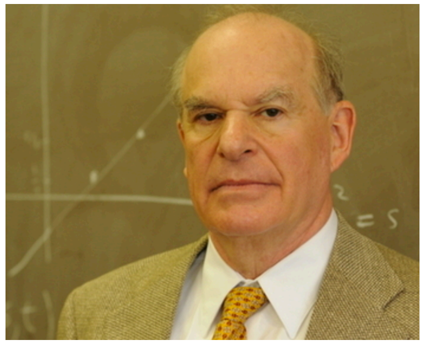


Lesson #1: If a theory developed for purpose A turns out to be better suited for purpose B, modify your goal accordingly.

The original goal of string theory was a theory of hadrons, but it turned out to work better as a theory of quantum gravity and unification. The massless particles should be identified as gauge particles and a graviton rather than vector mesons and a Pomeron.

When Yang and Mills formulated gauge theory in 1954, they identified SU(2) gauge fields with ρ mesons. 15 years later theorists developing dual models (the original name of string theory) made the same "mistake".

In 1974 we proposed to change the goal of string theory. It took another decade for the advantages of this interpretation to be widely appreciated. Perhaps there is a lesson in that, as well.



Lesson #3: **When working on hard problems explore generalizations with additional parameters.**

This lesson seems to be widely appreciated. There are many examples in the literature.

A couple of well-known examples are the Ω background for $\mathcal{N} = 2$ gauge theories and the $\mathbb{Z}_k$ orbifold generalization of $AdS_4 \times S^7$, which plays an important role in ABJM theory.