Non-perturbative rheological behavior of a far-from-equilibrium expanding plasma

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Collaboration with

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"Non-perturbative rheological behavior of a far-from-equilibrium expanding plasma", [arXiv 1805.087771]
"Dynamical systems and nonlinear transient rheology of the far-from-equilibrium Bjorken flow", in Preparation

"Physics of Nonequilibrium systems"

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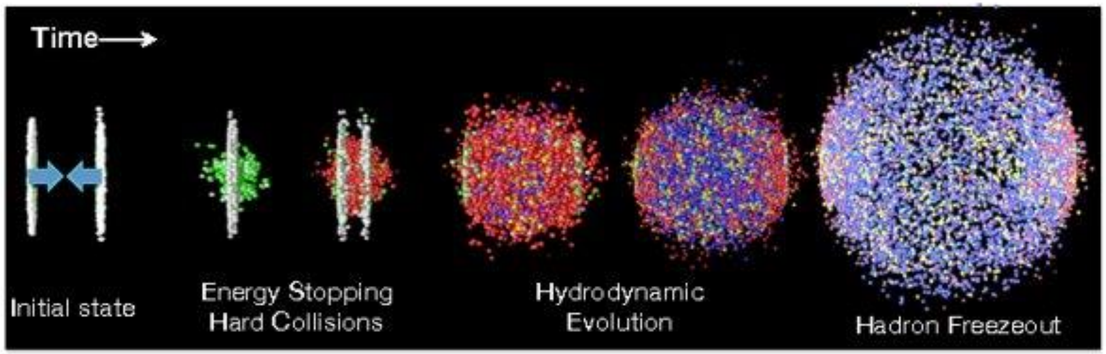
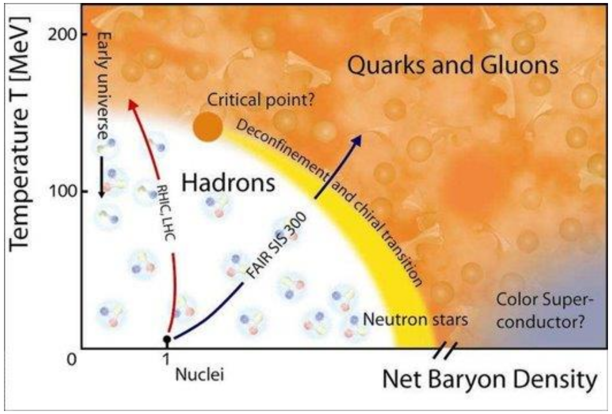
Introduction: motivation





- QCD and Heavy-ion collision
  - Described by quarks and gluon
- QGP and early universe --- Thermal process in time evolution
- **Bjorken flow** (Kinetic theory, RTA approximation)

$$\partial_\tau f(\tau, p_T, p_\varsigma) = -\frac{1}{\tau_r(\tau)}\left[f(\tau, p_T, p_\varsigma) - f_{eq.}(-u \cdot p/T)\right]$$

- Connection to Hydrodynamics
- Renormalized transport coefficients

Introduction: Bjorken flow - RTA approximation -

$$\frac{d^6 N(t,\mathbf{x},\mathbf{p})}{d^3\mathbf{x}d^3\mathbf{p}} = f(t,\mathbf{x},\mathbf{p}), \quad p^\mu \partial_\mu f(t,\mathbf{x},\mathbf{p}) = C(t,\mathbf{x},\mathbf{p})$$

[Bhatnagar et al. 54, Anderso et al. 74]

- Collision of Kernel $C(t,\mathbf{x},\mathbf{p})$: input from a micro theory
- RTA approximation (relativistic massless particle)

$$\partial_\tau f(\tau, p_T, p_\varsigma) = -\frac{1}{\tau_r(\tau)}\left[f(\tau, p_T, p_\varsigma) - f_{eq.}(-u\cdot p/T)\right] \qquad \tau_r = \frac{\theta_0}{T(\tau)} \qquad \theta_0 = 5\eta_0/s$$

$$g_{\mu\nu} = \mathrm{diag.}(-1,1,1,\tau^2) \qquad \tau = \sqrt{t^2 - z^2} \qquad \varsigma = \mathrm{arctanh}(z/t) \qquad u^2 = -1$$

  - Depends on Milne time and momentum
  - Boost + 2D isometries + parity (z-axis)
    - Bjorken flow : $ISO(2)\times SO(1,1)\times Z_2$ (on $\mathcal{M}_4$) **[Bjorken 83]** **Today's talk**
    - Gubser flow : $SO(3)\times SO(1,1)\times Z_2$ (on dS_4) **[Gubser 10]**

- Integration form

$$f(\tau, p_T, p_\varsigma) = D(\tau,\tau_0)\, f_0(\tau_0, p_T, p_\varsigma) + \int_{\tau_0}^{\tau} \frac{d\tau'}{\tau_r(\tau')} D(\tau,\tau')\, f_{eq.}(\tau', p_T, p_\varsigma),$$

$$D(\tau_2,\tau_1) = \exp\left[-\int_{\tau_1}^{\tau_2} d\tau'/\tau_r(\tau')\right], \quad f_0(\tau_0, p_T, p_\varsigma) = \exp\left[\frac{\sqrt{p_T^2 + (1+\xi_0)(p_\varsigma/\tau_0)^2}}{\Lambda_0}\right]$$

Initial constants (τ_0, T_0, ξ_0)

Introduction: BE to Hydrodynamics

- Gradient expansion ⇨ Hydrodynamics

$$p^\mu \partial_\mu f_{\mathbf{p}} = -\frac{T^2}{C_{\mathbf{p}}}(f - f_{eq.}) \,, \qquad f_{\mathbf{p}} = f_{eq.} + \alpha\,\delta f_1 + \alpha^2\,\delta f_2 + \ldots$$

[Chapman, Enskog]

$$C_{\mathbf{p}} = -T^2\tau_r/(u\cdot p) \text{ with } \tau_r = \theta_0/T$$

$$\mathcal{O}(\alpha): \quad \delta f_1 = C_{\mathbf{p}}\, p^\mu \partial_\mu f_{eq.}\,,$$
$$\mathcal{O}(\alpha^2): \quad \delta f_2 = C_{\mathbf{p}}\, p^\mu \partial_\mu \delta f_1\,,$$

$$\delta f = \chi_p C_p \left[-\tilde{p}^2 \frac{4}{9T^4\tau^2} - \tilde{p}\frac{20}{9T^2\tau^2} - \tilde{p}^2\frac{4}{9T^2\tau^2} - \tilde{p}^3\frac{4}{9T^2\tau^2} - \tilde{p}^5\frac{64}{315T^2\tau^2} + \tilde{p}^6\frac{16}{315T^2\tau^2} + \cdots \right] P_0(\cos\theta)$$
$$+ \left[-\tilde{\chi}_p\tilde{p}^2\left(\frac{2}{3\tau T}\right) + \tilde{\chi}_p'\tilde{C}_p\tilde{p}^4\left(\frac{8}{63\tau^2T^2}\right) - \tilde{\chi}_p\tilde{C}_p\tilde{p}^3\left(\frac{8}{9\tau^2T^2}\right) + \cdots \right] P_2(\cos\theta) \; + \cdots$$

$$\tilde{\chi}_p = -C_p f_{eq}' \qquad \tilde{p} = p/T$$

$$\boxed{w = \tau T(\tau)}$$

$$\eta = \frac{2}{15T^3}\int_p p^4 \chi_{0,p} = \frac{4\tau_R T^4}{5\pi^2}$$
$$\eta\tau_\pi = \frac{2}{15T^5}\int_p p^5 \chi_{2,p} = \frac{4\tau_R^2 T^4}{5\pi^2}$$
$$\lambda_1 = \frac{8}{105T^6}\int_p p^6 \chi_{1,p} - \eta\tau_\pi = \frac{12\tau_R^2 T^4}{35\pi^2}$$

[Bhalerao et al. 14]

Introduction: Mathematics

- **Dynamical system** (Non-autonomous ODE)
- Analytic method
- **Transseries analysis** (classical asymptotics)

  Generalization of asymptotic expansion
- Connection to other math tools:
  - Resurgence theory
  - Bifurcation theory
  - Conley index theory
  - Etc...
- Basic tools for application to other physical system
  - Non-relativistic system
  - Rheology
  - Non-perturbative RG eq.
  - Etc...

Contents

- Introduction
- Transseries analysis for Bjorken flow
    - Boltzmann equation to dynamical system
    - Transseries analysis
- Global structure of the dynamical system
    - Phase portrait
    - Initial value problems
- Conclusion

Transseries analysis for Bjorken flow

Boltzmann equation (Bjorken flow)

$$\partial_\tau f\left(\tau, p_T, p_\varsigma\right) = -\frac{1}{\tau_r(\tau)}\left[f\left(\tau, p_T, p_\varsigma\right) - f_{eq.}\left(-u\cdot p/T\right)\right],$$

Chapman-Enskog expansion

Moment expansion

Hydrodynamics

EM tensor
Transport coeffs

Navier-Stokes limit (?)

Dynamical system
(Non-autonomous system)

Transseries Analysis
(Resurgence)

Reduction to Dynamical System

- Moment expansion $ISO(2) \times SO(1,1) \times Z_2$

$$f(\tau, p_T, p_\varsigma) = f_{eq.}\left(\frac{p^\tau}{T}\right)\left[\sum_{n=0}^{N_n}\sum_{l=0}^{N_l} c_{nl}(\tau)\,\mathscr{P}_{2l}\left(\frac{p_\varsigma}{\tau p^\tau}\right)\mathscr{L}_n^{(3)}\left(\frac{p^\tau}{T}\right)\right]$$

Legendre polynomial $\mathcal{P}_l(x)$

Laguerre polynomial $\mathcal{L}_n^{(3)}(x)$

$f_{eq.}(x) = e^{-x}$

[Grad 49, Romatschke et al. 11]

$$c_{nl}(\tau) = 2\pi^2\frac{(4l+1)}{T^4(\tau)}\frac{\Gamma(n+1)}{\Gamma(n+4)}\left\langle (p^\tau)^2 P_{2l}\left(\frac{p_\varsigma}{\tau p^\tau}\right)\mathscr{L}_n^{(3)}\left(\frac{p^\tau}{T}\right)\right\rangle$$

$$\langle \mathscr{O}\rangle_X \equiv \int_{\mathbf{p}} \mathscr{O}(x^\mu, p^\mu) f_X(x^\mu, p_i) \text{ with } \int_{\mathbf{p}} \equiv \int d^2p_T dp_\varsigma / [(2\pi)^3 \tau p^\tau]$$

- EM tensor $T^{\mu\nu} = \langle p^\mu p^\nu \rangle = \varepsilon u^\mu u^\nu + P_L l^\mu l^\mu + P_T \Xi^{\mu\nu}$ [Molnar et al. 16]

$$\varepsilon = \langle(-u\cdot p)^2\rangle = \frac{3}{\pi^2}c_{00}T^4, \quad P_T = \left\langle \frac{1}{2}\Xi^{\mu\nu}p_\mu p_\nu\right\rangle = \varepsilon\left(\frac{1}{3} - \frac{1}{15}c_{01}\right), \quad P_L = \left\langle (l\cdot p)^2\right\rangle = \varepsilon\left(\frac{1}{3} + \frac{2}{15}c_{01}\right)$$

$$\bar{\pi} = \frac{2}{3}\left(\frac{P_L - P_T}{\varepsilon}\right) = \frac{2}{15}c_{01}$$

$$\boxed{c_1 = -\frac{40}{3}w^{-1}(\eta/s)_0 - \frac{80}{9}Tw^{-2}(\tau_{\pi,0}(\eta/s)_0 - (\lambda_1/s)_0) + \dots .}$$

Reduction to Dynamical System

- Dynamical System (Non-linear ODE)

$$\frac{dc_{nl}}{d\tau}+\frac{1}{\tau}\left[\alpha_{nl}\,c_{nl+1}+\beta_{nl}\,c_{nl}+\gamma_{nl}\,c_{nl-1}-n\left(\rho_l\,c_{n-1l+1}+\psi_l\,c_{n-1l}+\phi_l\,c_{n-1l-1}\right)\right]$$
$$+\frac{1}{\tau_r(\tau)}\left(c_{nl}-\delta_{n,0}\delta_{l,0}\right)=0\,,\qquad c_{00}=1$$

$$\frac{1}{T}\frac{dT}{d\tau}+\frac{1}{3\tau}=-\frac{c_{01}}{30\,\tau}\,,$$

From E conservaiton law $\mathfrak{D}_\mu T^{\mu 0}=0$

$$\alpha_{nl}=\frac{(2+2l)(1+2l)(n+1-2l)}{(4l+3)(4l+5)}\,,\qquad \rho_l=\frac{(2l+1)(2l+2)}{(4l+3)(4l+5)}\,,$$
$$\beta_{nl}=\frac{2l(2l+1)(5+2n)}{3(4l+3)(4l-1)}-\frac{(4+n)}{30}c_{01}\,,\qquad \psi_l=\frac{1}{3}\left(\frac{4l(2l+1)}{(4l+3)(4l-1)}\right)-\frac{c_{01}}{30}\,,$$
$$\gamma_{nl}=(2l+2+n)\frac{(2l)(2l-1)}{(4l-3)(4l-1)}\,,\qquad \phi_l=\frac{(2l)(2l-1)}{(4l-3)(4l-1)}\,,$$

Reduction to Dynamical System

- Dynamical System (Non-linear ODE)

$$\boxed{w = \tau T(\tau)}$$

$$\frac{dc_{nl}}{dw} = -\frac{1}{1-\frac{c_{01}}{20}}\left[\frac{3}{2w}\left(\alpha_{nl}c_{nl+1} + \beta_{nl}c_{nl} + \gamma_{nl}c_{nl-1} - n\left(\rho_l c_{n-1l+1} + \psi_l c_{n-1l} + \phi_l c_{n-1l-1}\right) + \frac{3c_{nl}}{2\theta_0}\right)\right]$$

$$\mathscr{A}(w) = d\log T/d\log\tau = -\frac{1}{3}\left(\frac{c_{01}}{10}+1\right)$$

$$\alpha_{nl} = \frac{(2+2l)(1+2l)(n+1-2l)}{(4l+3)(4l+5)}, \qquad \rho_l = \frac{(2l+1)(2l+2)}{(4l+3)(4l+5)},$$

$$\beta_{nl} = \frac{2l(2l+1)(5+2n)}{3(4l+3)(4l-1)} - \frac{(4+n)}{30}c_{01}, \qquad \psi_l = \frac{1}{3}\left(\frac{4l(2l+1)}{(4l+3)(4l-1)}\right) - \frac{c_{01}}{30},$$

$$\gamma_{nl} = (2l+2+n)\frac{(2l)(2l-1)}{(4l-3)(4l-1)}, \qquad \phi_l = \frac{(2l)(2l-1)}{(4l-3)(4l-1)},$$

Transseries Analysis

[Ecalle 81-85, Costin 98]

- Generalization of asymptotic expansion

$$\boxed{F(w) = \sum_{k=1}^{\infty} a_k w^{-k} \Rightarrow \sum_{n=0}^{\infty}\sum_{k=1}^{\infty} \sigma^n e^{-nSw} a_{nk} w^{-k}}$$

- Divergent series (Radius of convergence = 0)
    - Factorial growth: $a_k \to AS^{-k}k! \quad \text{as} \quad k \to +\infty$
      (**singularities** on the positive real axis in the Borel plane)
    - Signal of existence of higher level trans-monomials
    - Resurgence relation
- Costin's formula $$\boxed{\frac{d\mathbf{c}}{dw} = -\left[\hat{\Lambda}\mathbf{c} + \frac{1}{w}(\mathfrak{B}\mathbf{c} + \mathbf{A})\right] + O(\mathbf{c}^2, \mathbf{c}/w^2),}$$
    - $w \to \infty, \mathbf{c} \to 0$ (converge into an IR fixed pt.)
    - **Transseries ansatz** and coefficients are **uniquely** determined.
      (up to normalization of integration consts)
    - Imaginary ambiguity cancellation works (if you want).

Evolution equation

$$\frac{d\mathbf{c}}{dw} = \mathbf{f}(w,\mathbf{c}), \quad \mathbf{f}(w,\mathbf{c}) = -\frac{1}{1-\frac{c_1}{20}}\left[\hat{\mathbf{A}}\mathbf{c} + \frac{1}{w}(\mathfrak{B}\mathbf{c} + c_1\mathfrak{D}\mathbf{c} + \mathbf{A})\right]$$

Transseries ansatz

$$\tilde{c}_i(w) = \sum_{|\mathbf{m}|\geq 0}^{\infty} \sum_{k=0}^{\infty} \tilde{u}_{i,k}^{(\mathbf{m})} E_k^{(\mathbf{m})}(w), \qquad \zeta^{\mathbf{m}}(w) = e^{-(\mathbf{m}\cdot\mathbf{S})w} w^{\mathbf{m}\cdot\tilde{\mathbf{b}}}$$

$$E_k^{(\mathbf{m})}(w) = \sigma^{\mathbf{m}} \zeta^{\mathbf{m}}(w) w^{-k}, \qquad \sigma^{\mathbf{m}} = \prod_{i=1}^{I} \sigma_i^{m_i},$$

$$\mathbf{S} \in \mathbb{R}_+^I, \quad \tilde{\mathbf{b}} \in \mathbb{C}^I, \quad \mathbf{m} \in \mathbb{N}_0^I$$

σ_i : integration consts

$$\tilde{\mathbf{c}} = U\mathbf{c}, \quad \tilde{\mathbf{A}} = U\mathbf{A}, \quad \hat{\mathfrak{B}} = U\mathfrak{B}U^{-1} = \mathrm{diag}(b_1,\cdots,b_I) \in \mathbb{C}^I, \quad \tilde{\mathfrak{D}} = U\mathfrak{D}U^{-1}$$

Substitute Boltzmann eq.

$$20\left[\left(\mathbf{m}\cdot\tilde{\mathbf{b}} + b_i - k\right)\tilde{u}_{i,k}^{(\mathbf{m})} + \left(\frac{3}{2\theta_0} - \mathbf{m}\cdot\mathbf{S}\right)\tilde{u}_{i,k+1}^{(\mathbf{m})}\right] + 20\tilde{A}_i\,\delta_{k,0}\delta_{\mathbf{m},\mathbf{0}}$$
$$-\sum_{|\mathbf{m}'|\geq 0}^{\mathbf{m}}\left[\sum_{k'=0}^{k}\left(\mathbf{m}'\cdot\tilde{\mathbf{b}} - k'\right) u_{1,k-k'}^{(\mathbf{m}-\mathbf{m}')}\tilde{u}_{i,k'}^{(\mathbf{m}')} - 20\sum_{i'=1}^{I}\sum_{k'=0}^{k} u_{1,k-k'}^{(\mathbf{m}-\mathbf{m}')}\tilde{\mathfrak{D}}_{ii'}\tilde{u}_{i',k'}^{(\mathbf{m}')} - \mathbf{m}'\cdot\mathbf{S}\sum_{k'=0}^{k+1} u_{1,k-k'+1}^{(\mathbf{m}-\mathbf{m}')}\tilde{u}_{i,k'}^{(\mathbf{m}')}\right] = 0,$$

$$\Rightarrow \quad S_i = \frac{3}{2\theta_0}, \qquad \tilde{b}_i = -\left(b_i - \frac{1}{5}\right)$$

Recursively solve order by order.

Transasymptotic matching [Basar et al. 15]

- RG equation of transport coefficients

$$\tilde{c}_i(w) = \sum_{k\geq 0}^{\infty} \tilde{C}_{i,k}(\sigma\zeta(w))w^{-k}$$

$$\hat{\zeta}_i := \partial/\partial \log(\sigma_i\zeta_i)$$

$$\boxed{\begin{aligned}&20\left[\left((\tilde{\mathbf{b}}\cdot\hat{\zeta}-k)+b_i\right)\tilde{C}_{i,k}-\mathbf{S}\cdot\hat{\zeta}\tilde{C}_{i,k+1}+\frac{3}{2\theta_0}\tilde{C}_{i,k+1}\right]+20\tilde{A}_i\delta_{k,0}\\&-\sum_{k'=0}^{k}C_{1,k-k'}\left(\tilde{\mathbf{b}}\cdot\hat{\zeta}-k'\right)\tilde{C}_{i,k'}+20\sum_{i'=1}^{I}\sum_{k'=0}^{k}C_{1,k-k'}\tilde{\mathfrak{D}}_{ii'}\tilde{C}_{i',k'}+\sum_{k'=0}^{k+1}C_{1,k-k'+1}\mathbf{S}\cdot\hat{\zeta}\tilde{C}_{i,k'}=0,\end{aligned}}$$

- Simultaneous PDE ⇒ ODE and solvable if L=1, N=0

$$C_{1,0}(\sigma\zeta) = -20W_\zeta,$$

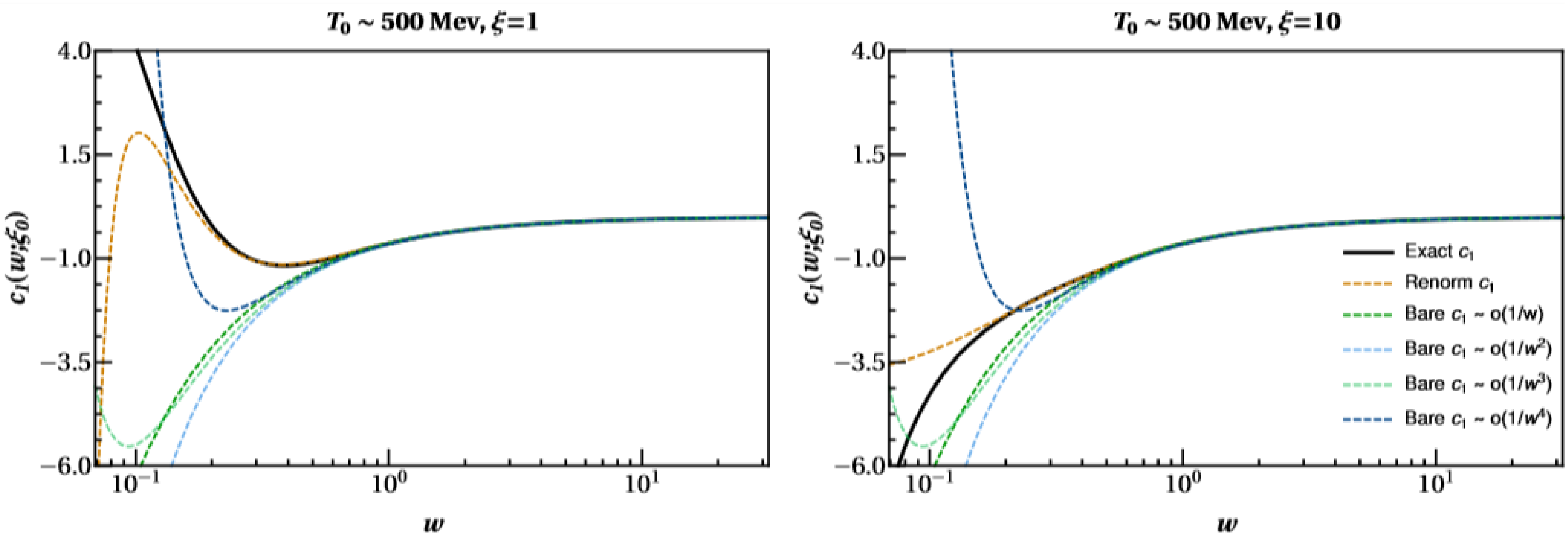
$W_\zeta := W(-\sigma\zeta/20)$ and the Lambert W function.

$$C_{1,1}(\sigma\zeta) = -\frac{8\theta_0\left(50W_\zeta^3+105W_\zeta^2+36W_\zeta+5\right)}{15\left(W_\zeta+1\right)},$$

$$C_{1,2}(\sigma\zeta) = -\frac{8\theta_0^2}{7875(W_\zeta+1)}\left[\frac{25\left(700W_\zeta^4+2195W_\zeta^3+966W_\zeta^2+20\right)}{W_\zeta}+\frac{4032}{(W_\zeta+1)^2}+3685\right],$$

Comparison with the exact solution

L=1, N=0, O(1/w)



Deviation from the NS limit

- NS limit $\Rightarrow$ ~ 1/w

  EM tensor is related only with c_{01}

$$\varepsilon = \langle (-u\cdot p)^2 \rangle = \frac{3}{\pi^2} c_{00} T^4 , \quad P_T = \left\langle \frac{1}{2} \Xi^{\mu\nu} p_\mu p_\nu \right\rangle = \varepsilon \left(\frac{1}{3} - \frac{1}{15} c_{01} \right) , \quad P_L = \left\langle (l\cdot p)^2 \right\rangle = \varepsilon \left(\frac{1}{3} + \frac{2}{15} c_{01} \right)$$

- However … c_{11} has also the same asymptotics.

$$c_{nl} + \frac{2\theta_0}{3w} A_{nl} + \mathscr{O}(1/w^2) = 0 \quad \text{for} \quad (n,l) = (0,1), (1,1). \qquad c_{01}, c_{11} \sim 1/w$$

- Deviation from the NS hydro should exists in ~ 1/w due to c_{11} !!

Deviation from the NS limit

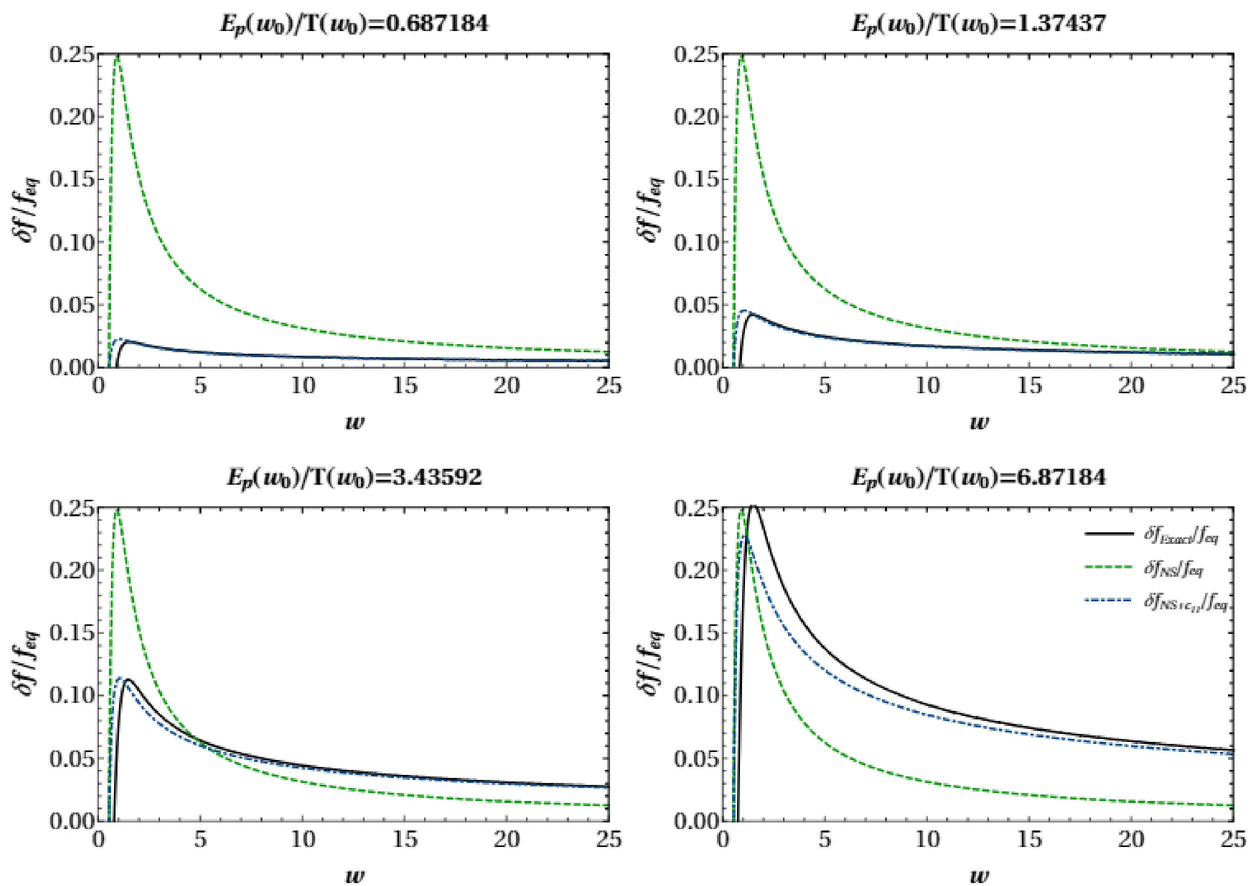
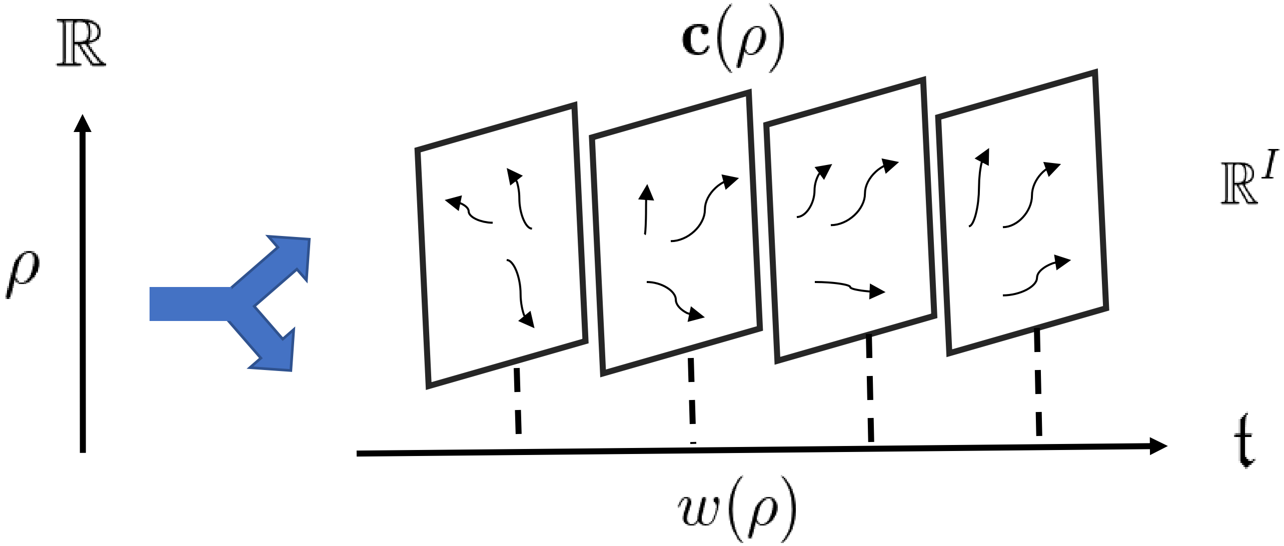


FIG. 7: Deviation of distribution with respect to background f_{eq} at different initial energy. The black lines are computed by exact RTA. The green dot-dashed lines are constructed by c_{01} for ansatz up to leading asymptotic order (aka Navier-Stokes); the blue dashed lines are constructed by ansatz with the truncation scheme $N = 1$, $L = 1$ including c_{11} to leading order $O(1/w)$. The energy-dependent thermalization is only captured by the truncation to higher moments c_{nl} $n \geq 1$.

Global structure of the dynamical system

Phase portrait

- Trivial fiberization $\quad \mathcal{M} = \mathbb{R}^I \times \mathfrak{t}$
  - Base space : $\mathfrak{t}$
  - Fiber space : $\mathbb{R}^I$

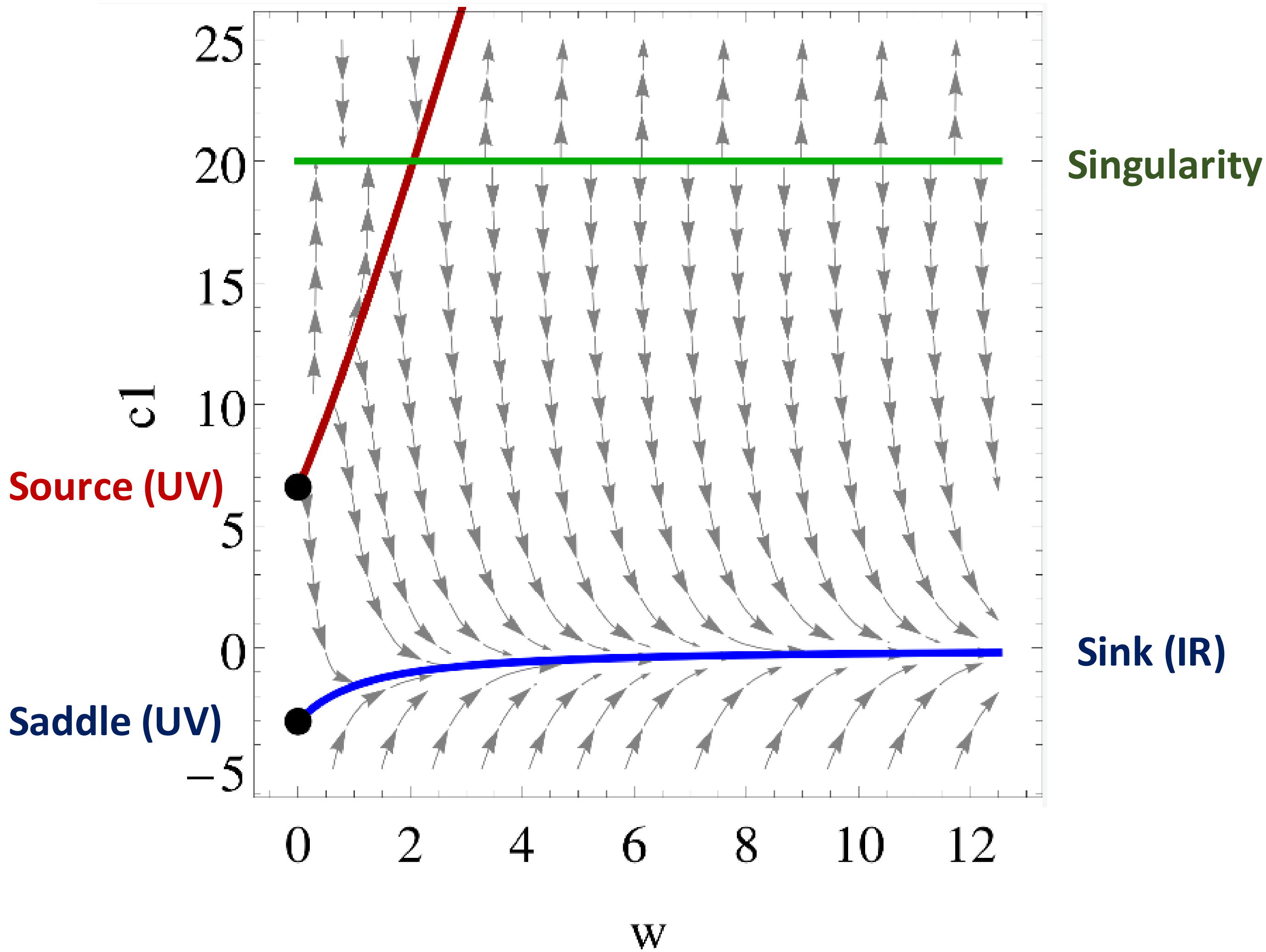


Skew product $\quad \dfrac{d\mathbf{c}}{d\rho} = \check{\mathbf{f}}(\mathbf{c}, w(\rho), \rho), \qquad \dfrac{dw}{d\rho} = g(\rho)$

"Time dependent control parameter" in Bifurcation theory

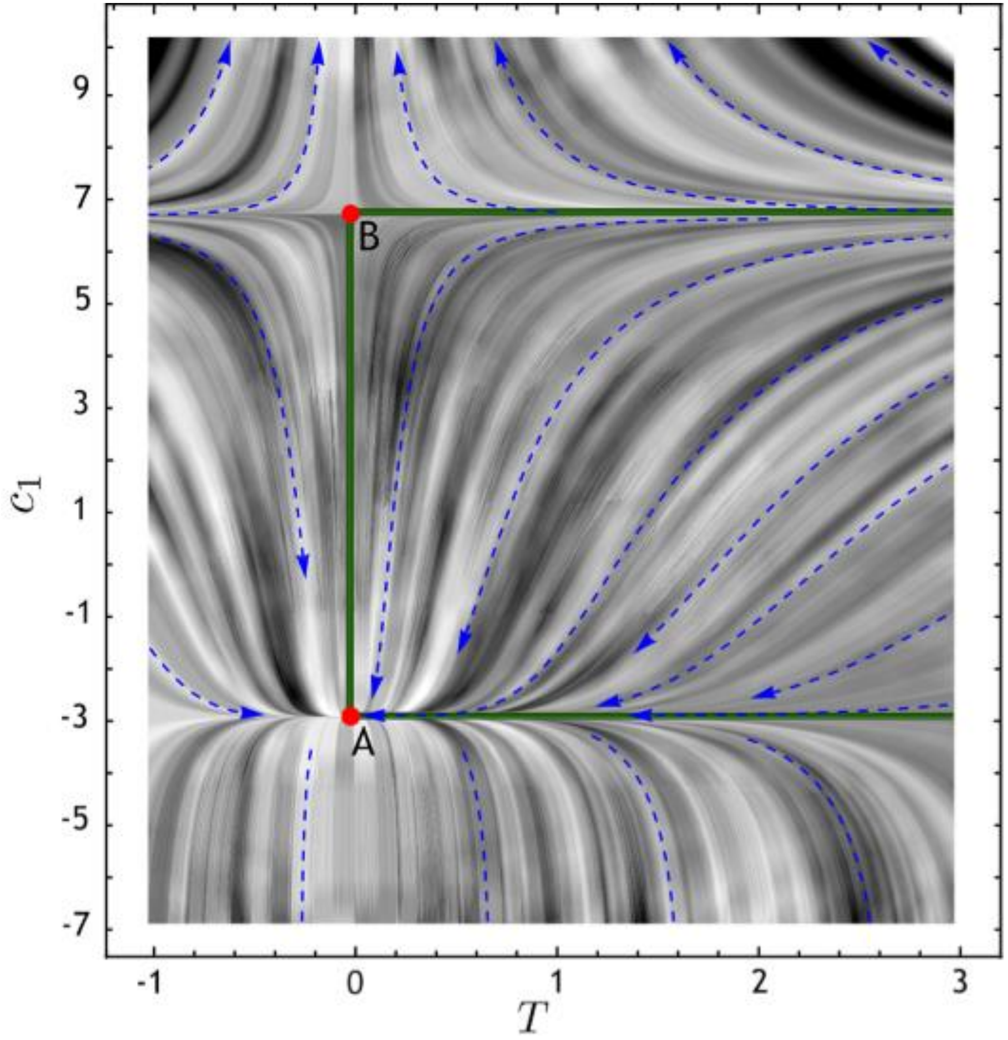
Phase portrait

$$\frac{d\mathbf{c}}{dw} = \mathbf{f}(w,\mathbf{c}), \quad \mathbf{f}(w,\mathbf{c}) = -\frac{1}{1-\frac{c_1}{20}}\left[\hat{\mathbf{A}}\mathbf{c} + \frac{1}{w}\left(\mathfrak{B}\mathbf{c} + c_1\mathfrak{D}\mathbf{c} + \mathbf{A}\right)\right]$$



Phase portrait

Near $\mathcal{T} = 0$



Initial value problem

- Integration form: (τ_0, T_0, ξ_0)

  ⇨ essentially ξ_0 in w coordinate
- Transseries: # of σ = $(N+1) \times (L+1) - 1$
- $\boldsymbol{\sigma}(\xi_0)$ gives a *one dimensional* orbit on $\boldsymbol{\sigma}$ space

$$\begin{aligned} \boldsymbol{\sigma} : [-1, +\infty) &\rightarrow \mathbb{R}^I \times \{w_0\} \simeq \mathscr{F}, \quad w_0 \in \mathbb{t} \\ \xi_0 &\mapsto \boldsymbol{\sigma}(\xi_0), \end{aligned}$$

where $\mathbb{t} = (0, +\infty)$ is the space of time and $\mathscr{F}$ is the space of integral curves

- Invariant subspace of the flow is *two dimensions*

Outstanding problem: How to make $\boldsymbol{\sigma}(\xi_0)$??

Conclusion

- Boltzmann equation $\Rightarrow$ **Dynamical system** of Bjorken flow via. moment expansion
- Application of **Transseries** to the dynamical system
  $\Rightarrow$ Beyond hydrodynamics
  Non-hydro mode can be uniquely determined.
- Renormalized transport coefficients
- Deviation from the NS hydro

Future work

- (Beyond) Linear response theory
- More realistic model $\Rightarrow$ space dependence $\Rightarrow$ PDE
- Condensed matter, non-relativistic system, ...