

Conformal symmetry in nonrelativistic systems and hydrodynamics

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基研研究会・iTHERMS研究会
「非平衡系の物理学－階層性と普遍性－」

December 26 - 28 (2018)

Plan of this talk

I would like to “review”
conformal symmetry in nonrelativistic systems,
its consequences on **(hydro)dynamics**,
and related experiments with **ultracold atoms**.

1. Schrödinger algebra
2. Breathing mode
3. Efimovian expansion
4. Bulk viscosity

1. Schrödinger algebra

Y. Nishida & D. T. Son, PRD (2007); arXiv:1004.3597
Y. Castin & F. Werner, PRA (2006); arXiv:1103.2851

Nonrelativistic CFT

Maximal spacetime symmetries of

$$S_{\text{free}} = \int dt d^d \vec{x} \psi^\dagger \left(i\partial_t + \frac{\vec{\nabla}^2}{2m} \right) \psi$$

U. Niederer, HPA (1972)

C. R. Hagen, PRD (1972)

- Translations in time and space
- Spatial rotations • Galilean boosts
- Scale transformation

$$\vec{x} \rightarrow e^{-s} \vec{x}, \quad t \rightarrow e^{-2s} t, \quad \psi \rightarrow e^{(d/2)s} \psi$$

- Conformal transformation

$$\vec{x} \rightarrow \frac{\vec{x}}{1 - ct}, \quad t \rightarrow \frac{t}{1 - ct},$$

$$\psi \rightarrow (1 - ct)^{d/2} \exp \left(i \frac{c}{1 - ct} \frac{m}{2} \vec{x}^2 \right) \psi$$

Nonrelativistic CFT

- Scale transformation

$$\vec{x} \rightarrow e^{-s}\vec{x}, \quad t \rightarrow e^{-2s}t, \quad \psi \rightarrow e^{(d/2)s}\psi$$

- Conformal transformation

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Nonrelativistic CFT

- **Scale transformation (infinitesimal)**

$$\begin{aligned}\delta_s \psi &= s \left(\frac{d}{2} + \vec{x} \cdot \vec{\nabla} + 2t\partial_t \right) \psi \\ &= -is [D - 2tH, \psi]\end{aligned}$$

$$D \equiv \int d^d \vec{x} \, \vec{x} \cdot \psi^\dagger (-i \vec{\nabla}) \psi$$

- **Conformal transformation**

$$\vec{x} \rightarrow \frac{\vec{x}}{1 - ct}, \quad t \rightarrow \frac{t}{1 - ct},$$

$$\psi \rightarrow (1 - ct)^{d/2} \exp \left(i \frac{c}{1 - ct} \frac{m}{2} \vec{x}^2 \right) \psi$$

Nonrelativistic CFT

- **Scale transformation (infinitesimal)**

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$$D \equiv \int d^d \vec{x} \vec{x} \cdot \psi^\dagger (-i \vec{\nabla}) \psi$$

- **Conformal transformation (infinitesimal)**

$$\begin{aligned}\delta_c \psi &= c \left(i \frac{m}{2} \vec{x}^2 - t \frac{d}{2} - t \vec{x} \cdot \vec{\nabla} - t^2 \partial_t \right) \psi \\ &= -ic [C - tD + t^2 H, \psi]\end{aligned}$$

$$C \equiv \frac{m}{2} \int d^d \vec{x} \vec{x}^2 \psi^\dagger \psi$$

~ mean square radius
~ harmonic potential

Nonrelativistic CFT

- **Scale transformation (infinitesimal)**

$$\begin{aligned}\delta_s \psi &= s \left(\frac{d}{2} + \vec{x} \cdot \vec{\nabla} + 2t\partial_t \right) \psi \\ &= -is [D - 2tH, \psi] \equiv -is [D(t), \psi]\end{aligned}$$

$D \equiv \int d^d \vec{x} \vec{x} \cdot \psi^\dagger (-i \vec{\nabla}) \psi$  $[D, H] = 2iH$

$$\dot{D}(t) = 0$$

- **Conformal transformation (infinitesimal)**

$$\begin{aligned}\delta_c \psi &= c \left(i \frac{m}{2} \vec{x}^2 - t \frac{d}{2} - t \vec{x} \cdot \vec{\nabla} - t^2 \partial_t \right) \psi \\ &= -ic [C - tD + t^2 H, \psi] \equiv -ic [C(t), \psi]\end{aligned}$$

$C \equiv \frac{m}{2} \int d^d \vec{x} \vec{x}^2 \psi^\dagger \psi$  $[C, H] = iD$

$$\dot{C}(t) = 0$$

Nonrelativistic CFT

Generators (D, C, H) obey $SO(2,1)$ Lie algebra

$$[D, H] = 2iH, \quad [C, H] = iD, \quad [D, C] = -2iC$$

scale invariance

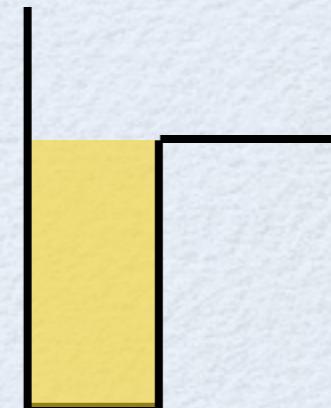
$$\dot{n} = -\vec{\nabla} \cdot \vec{j}$$

always true

$$H = H_0 + V(r) \rightarrow H' = H_0 + e^{-2s}V(e^{-s}r)$$



$$e^{-isD} H e^{isD} = e^{2s} H'$$



$H=H'$ for

- Inverse square [$V=1/r^2$] • 2D delta [$V=\delta^2(r)$] ??
- Zero-range ($r_0=0$) & infinite scattering length ($a=\infty$)

2. Breathing mode

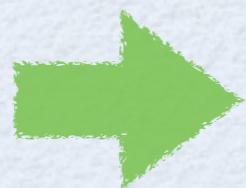
Y. Nishida & D. T. Son, PRD (2007); arXiv:1004.3597
Y. Castin & F. Werner, PRA (2006); arXiv:1103.2851

Operator-State correspondence

$$[D, H] = 2iH, \quad [C, H] = iD, \quad [D, C] = -2iC$$

Local operator \mathcal{O} has scaling dimension Δ when

$$[D, \mathcal{O}] = i\Delta \mathcal{O}$$



$$[D, [H, \mathcal{O}]] = i(\Delta + 2) [H, \mathcal{O}]$$

$$[D, [C, \mathcal{O}]] = i(\Delta - 2) [C, \mathcal{O}]$$

\mathcal{O} is primary when $[C, \mathcal{O}_{\text{pri}}] = 0$ $\left(C \equiv \frac{m}{2} \int d^d \vec{x} \vec{x}^2 \psi^\dagger \psi \right)$

$|\mathcal{O}\rangle = e^{-H/\omega} \mathcal{O}_{\text{pri}}^\dagger |0\rangle$ is eigenstate of

$$(H + \omega^2 C) |\mathcal{O}\rangle = e^{-H/\omega} \left[e^{H/\omega} (H + \omega^2 C) e^{-H/\omega} \right] \mathcal{O}_{\text{pri}}^\dagger |0\rangle$$

$$= e^{-H/\omega} (\omega^2 C - i\omega D) \mathcal{O}_{\text{pri}}^\dagger |0\rangle$$

$$\text{Hamiltonian} \quad = e^{-H/\omega} (\Delta\omega) \mathcal{O}_{\text{pri}}^\dagger |0\rangle = \Delta\omega |\mathcal{O}\rangle$$

with harmonic potential

energy eigenvalue

Operator-State correspondence

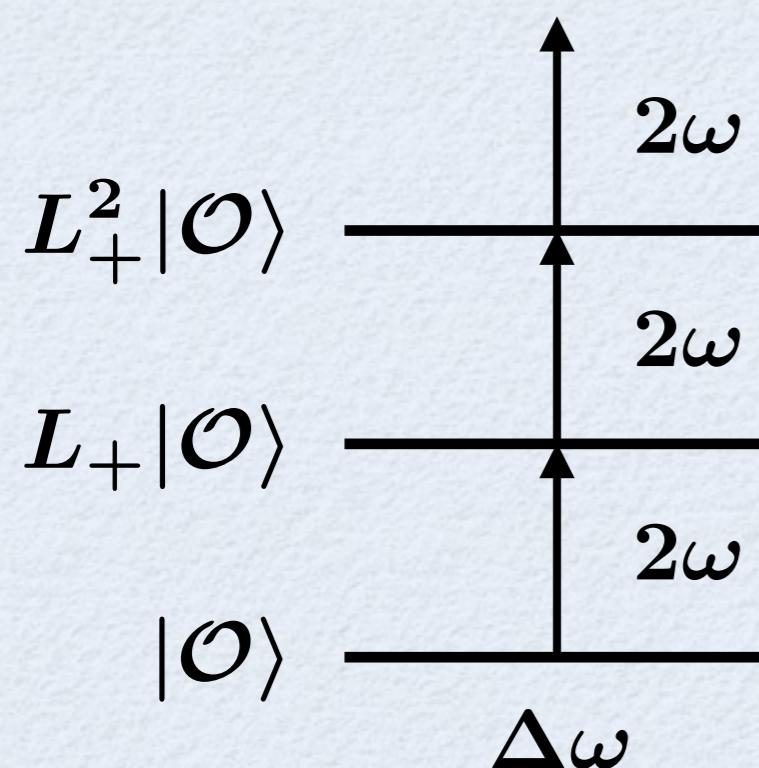
$$[D, H] = 2iH, \quad [C, H] = iD, \quad [D, C] = -2iC$$



$$H_\omega \equiv H + \omega^2 C, \quad L_\pm \equiv H - \omega^2 C \pm i\omega D$$

$$[H_\omega, L_\pm] = \pm 2\omega L_\pm, \quad [L_+, L_-] = -4\omega H_\omega$$

raising & lowering operators



$$L_- |\mathcal{O}\rangle = 0$$

$$H_\omega L_+^n |\mathcal{O}\rangle = (\Delta\omega + 2n\omega) L_+^n |\mathcal{O}\rangle$$

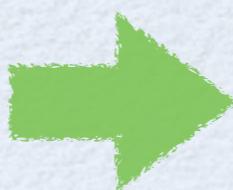
Valid for
any scale invariant systems
confined by harmonic potential

Breathing mode

Arbitrary time-evolving state $|\Psi_t\rangle = e^{-iH_\omega t}|\Psi_0\rangle$

$$\begin{aligned}\langle C \rangle &= \langle \Psi_0 | e^{iH_\omega t} \frac{2H_\omega - L_+ - L_-}{4\omega^2} e^{-iH_\omega t} |\Psi_0\rangle \\ &= \langle \Psi_0 | \frac{2H_\omega - e^{i2\omega t}L_+ - e^{-i2\omega t}L_-}{4\omega^2} |\Psi_0\rangle \\ &\equiv \frac{\langle \Psi_0 | H_\omega | \Psi_0 \rangle - \cos(2\omega t + \varphi) |\langle \Psi_0 | L_+ | \Psi_0 \rangle|}{2\omega^2}\end{aligned}$$

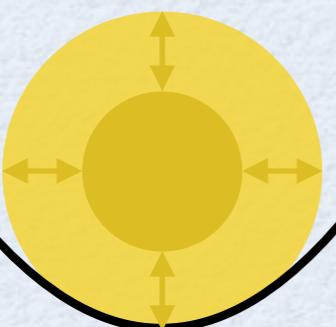
$$\left(C \equiv \frac{m}{2} \int d^d \vec{x} \, \vec{x}^2 \psi^\dagger \psi \right)$$



Mean square radius

$$\langle \vec{x}^2 \rangle = A + B \cos(2\omega t + \varphi)$$

**Undamped “breathing mode”
with frequency right at 2ω**



Breathing mode

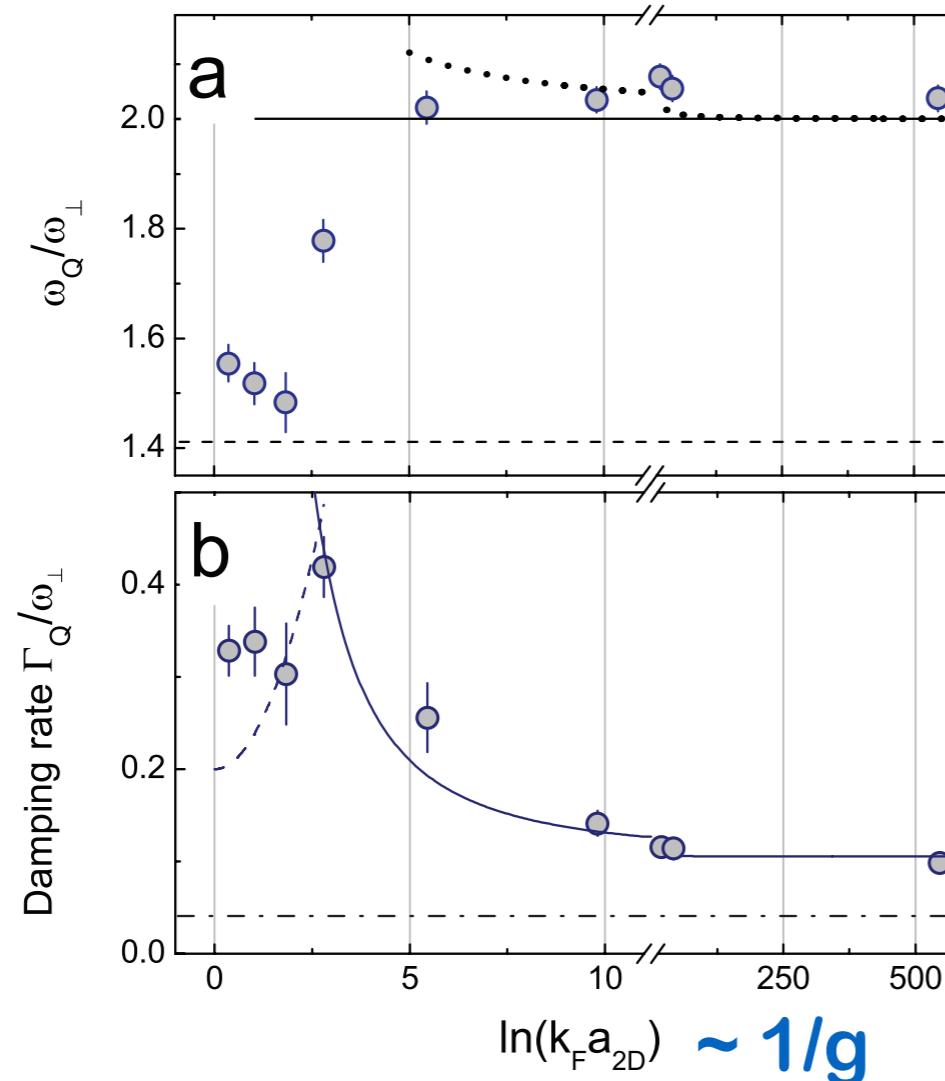
H is scale invariant for $V = -g \delta^2(\vec{r})$ in 2D ??

Tunable via Feshbach resonance
with ultracold atoms

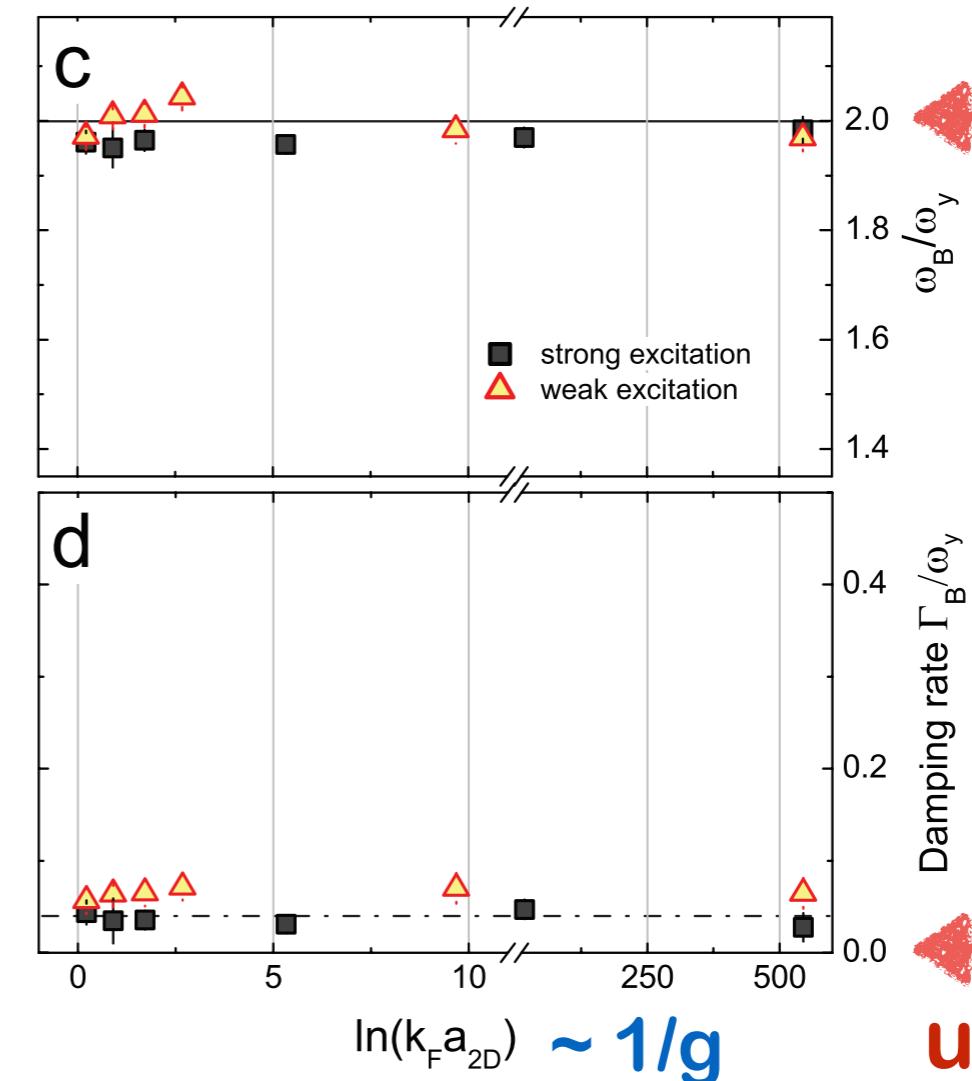
$T \sim 0.4 T_F$

M. Köhl's group, PRL (2012)

Quadrupole mode



Breathing mode



2ω

ω_B/ω_y

undamped

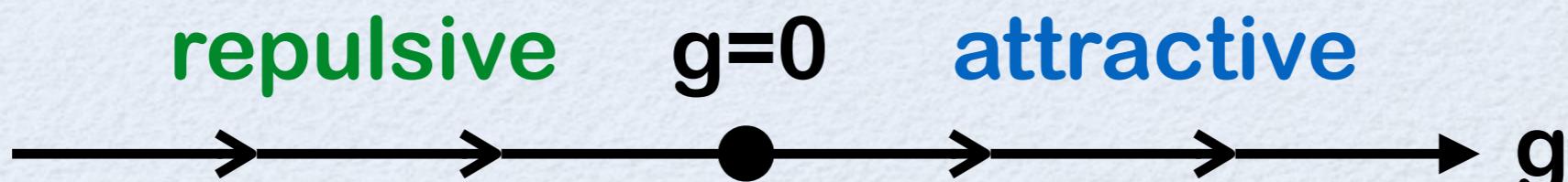
Conformal anomaly

H is scale invariant for $V = -g \delta^2(\vec{r})$ in 2D

only classically but **NOT** quantum mechanically

$$S = \int dt d^d \vec{x} \left[\psi^\dagger \left(i\partial_t + \frac{\vec{\nabla}^2}{2m} \right) \psi + \frac{g}{2} \psi^\dagger \psi^\dagger \psi \psi \right]$$

$$\frac{\partial g}{\partial \ln \Lambda} = \frac{m}{2\pi} g^2$$



quantum triviality

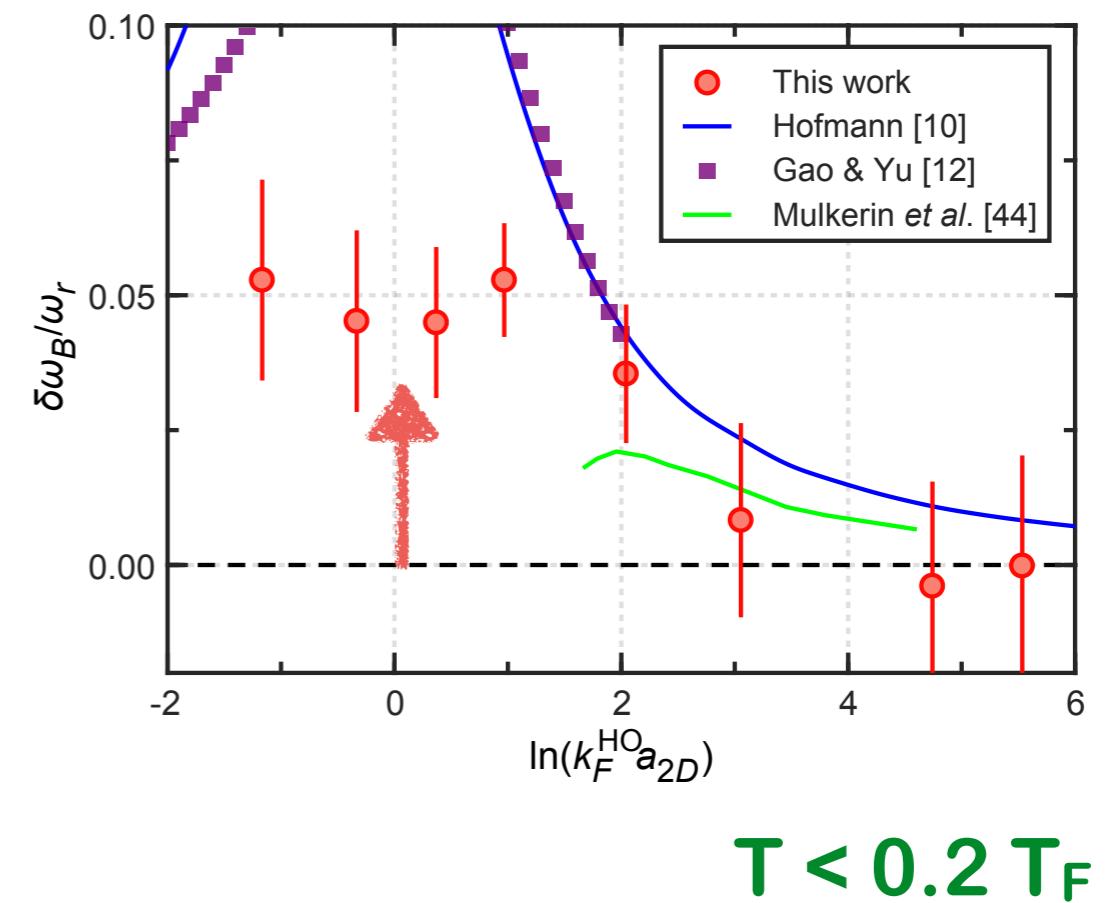
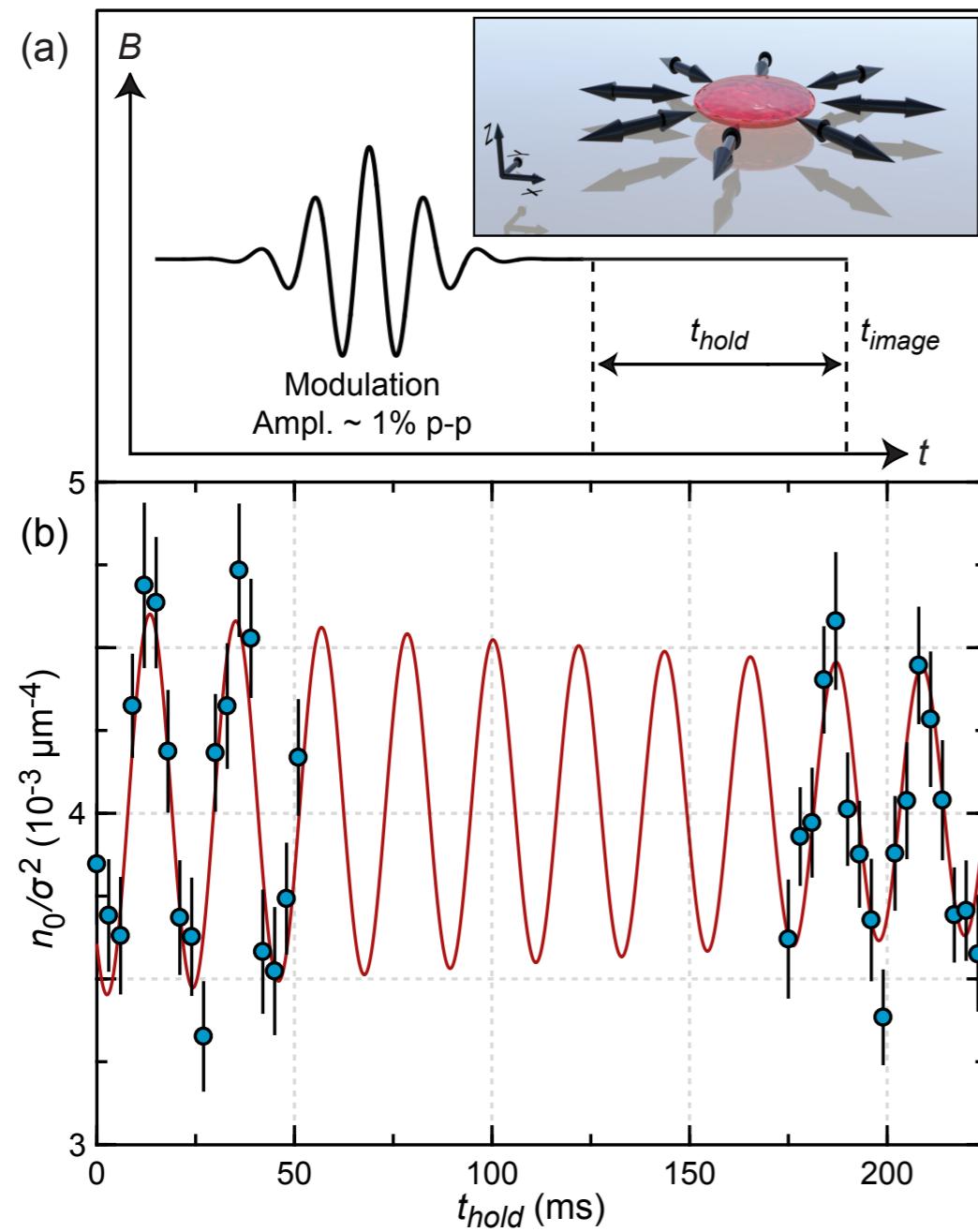
asymptotic freedom
⇒ Scale is generated

Cf. Conformal anomaly is insignificant at higher temperature

Conformal anomaly

H is scale invariant for $V = -g \delta^2(\vec{r})$ in 2D

only classically but **NOT** quantum mechanically

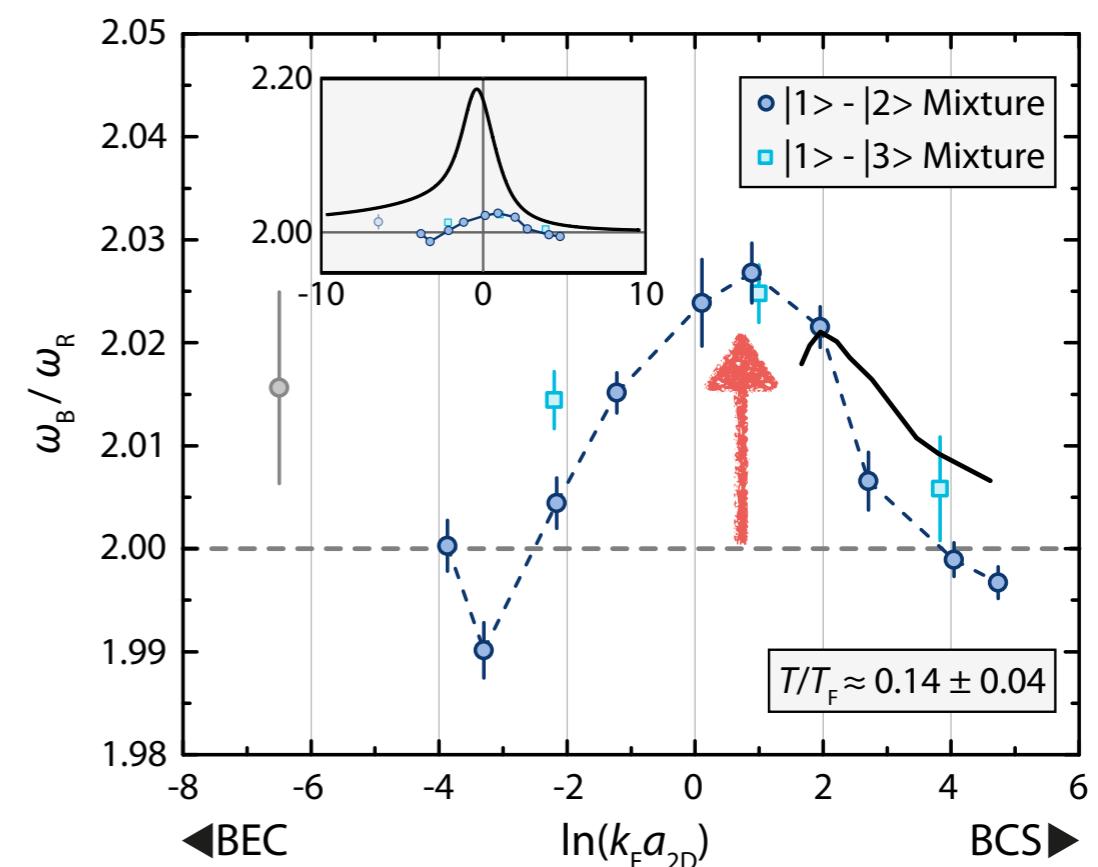
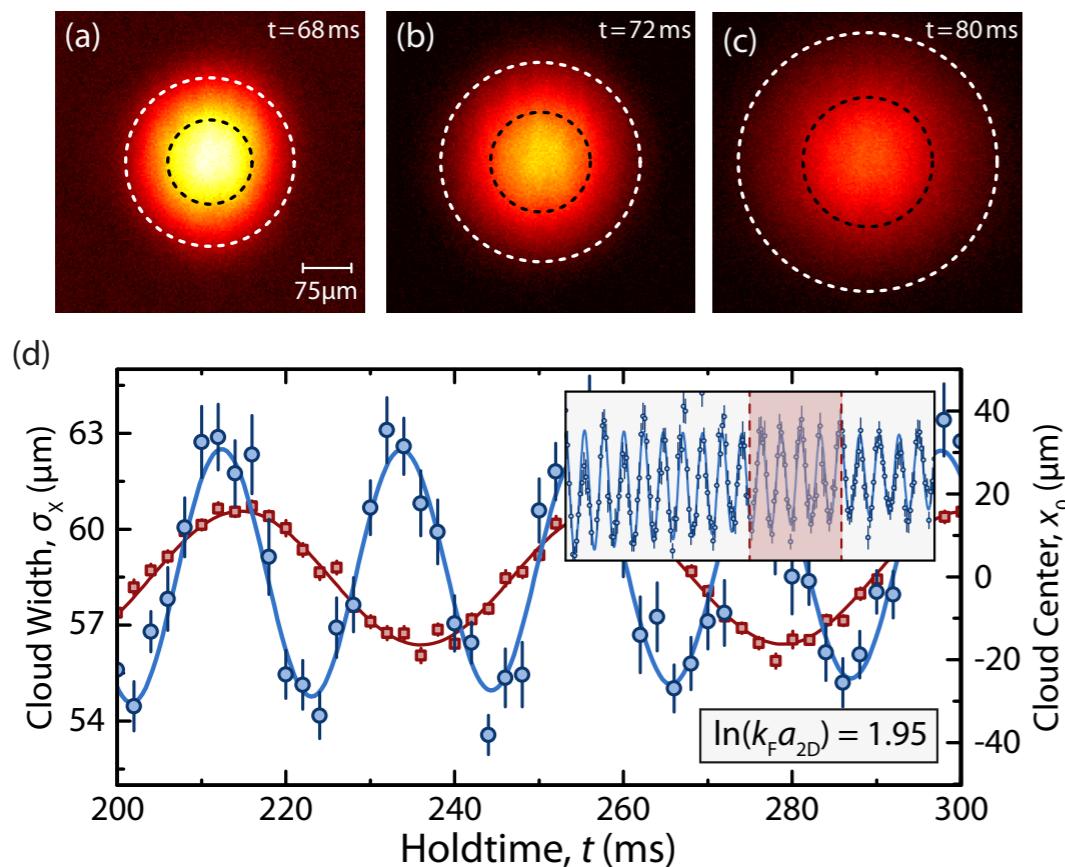


C. J. Vale's group, PRL (2018)

Conformal anomaly

H is scale invariant for $V = -g \delta^2(\vec{r})$ in 2D

only classically but **NOT** quantum mechanically



$T < 0.2 T_F$

S. Jochim's group, PRL (2018)

Cf. Small shift ($\sim 1\%$) compared to prediction ($\sim 10\%$) may be due to effective range effect

H. Hu et al., arXiv:1806.04383

3. Efimovian expansion

S. Deng et al., Science (2016); PRL (2018)
Z.-Y. Shi et al., PRA (2017)

Expanding harmonic potential

$H_\omega \equiv H + \omega^2 C$ is NOT scale invariant because

$$H_\omega \rightarrow e^{2s} [H + (e^{-2s}\omega)^2 C] \neq e^{2s} H_\omega$$

$$\left(\vec{x} \rightarrow e^{-s} \vec{x}, \quad t \rightarrow e^{-2s} t, \quad \psi \rightarrow e^{(d/2)s} \psi \right)$$

$H(t) \equiv H + (g/t)^2 C$ is scale invariant

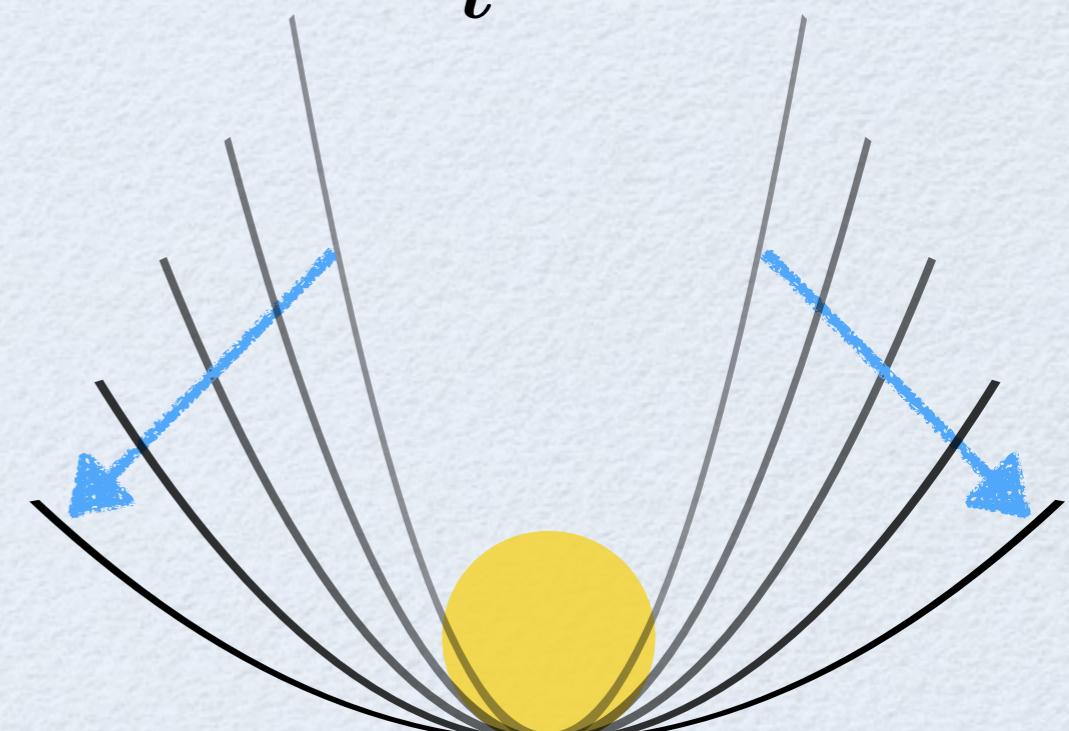
expanding harmonic potential

$$\omega = \frac{g}{t}$$

If scale invariance is intact

$$\langle \vec{x}^2 \rangle \sim t^\gamma$$

but it may be
anomalously broken ...
completely? or partially?



Mean square radius

Generators (D, C, H) obey $SO(2,1)$ Lie algebra

$$[D, H] = 2iH, \quad [C, H] = iD, \quad [D, C] = -2iC$$

Mean square radius for arbitrary time-evolving state

$$\langle \vec{x}^2 \rangle = (2/m) \langle \Psi_0 | U^\dagger(t) C U(t) | \Psi_0 \rangle$$

$$\frac{d}{dt} \langle C \rangle = i \langle [H(t), C] \rangle = \langle D \rangle$$

$$\frac{d^2}{dt^2} \langle C \rangle = i \langle [H(t), D] \rangle = 2 \langle H \rangle - \frac{2g^2}{t^2} \langle C \rangle$$

$$\frac{d^3}{dt^3} \langle C \rangle = 2i \langle [H(t), H] \rangle - \frac{d}{dt} \left(\frac{2g^2}{t^2} \langle C \rangle \right) = -\frac{4g^2}{t^2} \frac{d}{dt} \langle C \rangle + \frac{4g^2}{t^3} \langle C \rangle$$



$$t^3 \frac{d^3}{dt^3} \langle \vec{x}^2 \rangle + 4g^2 \left(t \frac{d}{dt} \langle \vec{x}^2 \rangle - \langle \vec{x}^2 \rangle \right) = 0$$

Mean square radius

$$t^3 \frac{d^3}{dt^3} \langle \vec{x}^2 \rangle + 4g^2 \left(t \frac{d}{dt} \langle \vec{x}^2 \rangle - \langle \vec{x}^2 \rangle \right) = 0$$

$$H + \left(\frac{g}{t} \right)^2 C$$

Scaling solution $\langle \vec{x}^2 \rangle \sim t^\gamma \Rightarrow \gamma = 1, 1 \pm \sqrt{1 - 4g^2}$

- $4g^2 < 1 \Rightarrow \langle \vec{x}^2 \rangle \sim t^{1+\sqrt{1-4g^2}}$ **scale invariant**

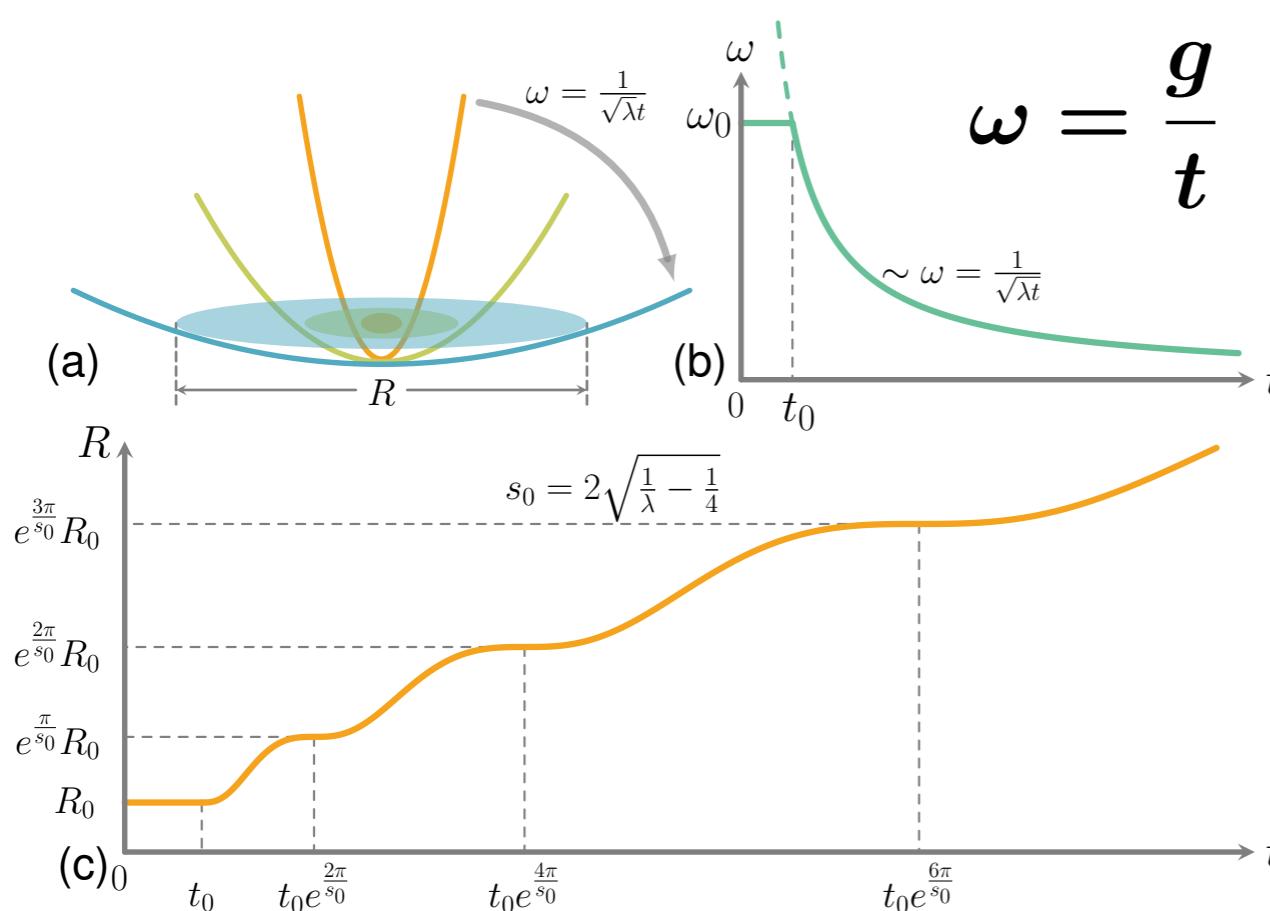
- $4g^2 > 1 \Rightarrow t^{1 \pm i\sqrt{4g^2-1}} = t e^{\pm i\sqrt{4g^2-1} \ln t}$

$$\langle \vec{x}^2 \rangle \sim t \left[A + B \cos \left(\sqrt{4g^2 - 1} \ln t + \varphi \right) \right]$$

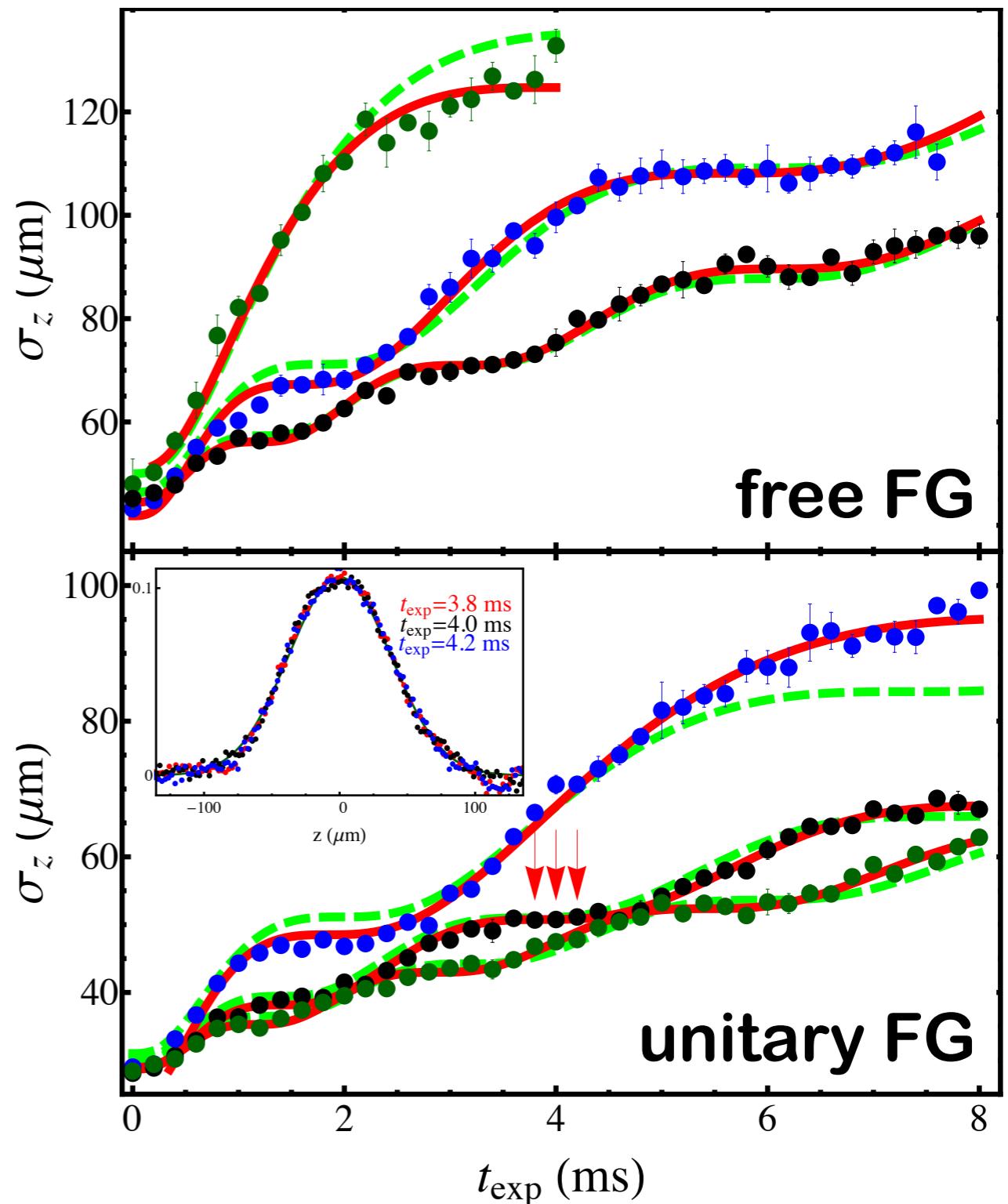
invariant under $t \rightarrow e^{2\pi n / \sqrt{4g^2-1}} t \quad (\forall n \in \mathbb{Z})$

Discrete scale invariance

Efimovian expansion



power law $\sim t$
+ log periodic oscillation

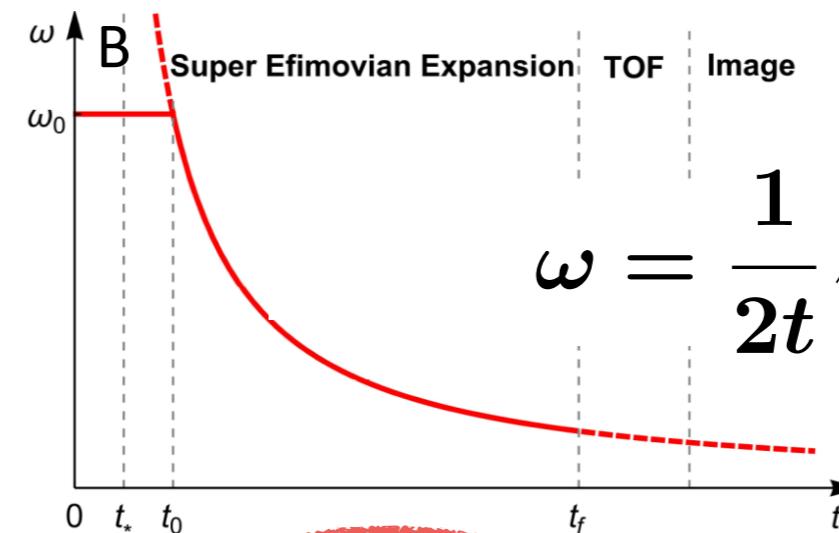
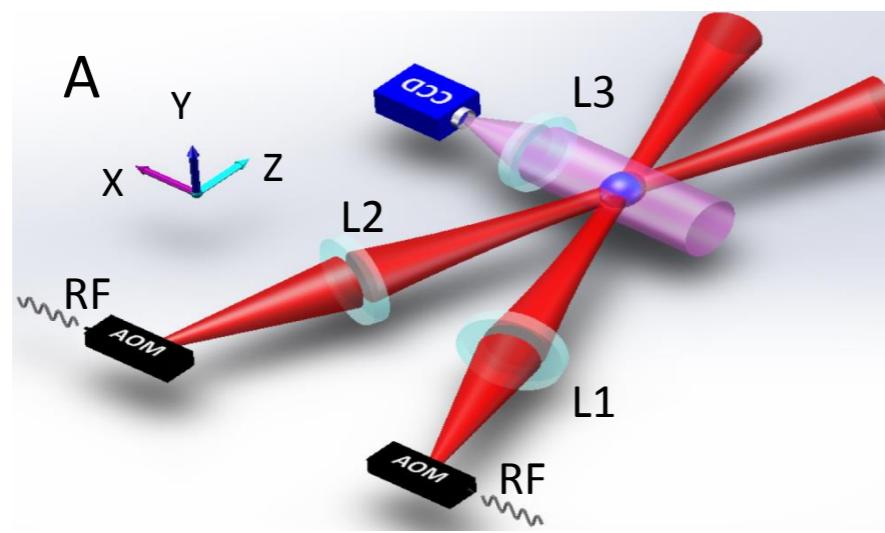


S. Deng et al., Science (2016)

Cf. Efimov effect (discrete scale inv. in 3-boson physics)

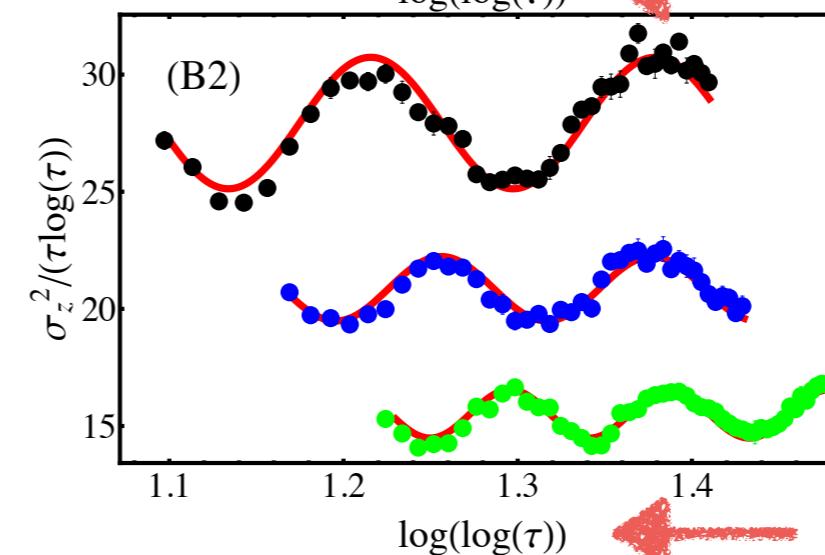
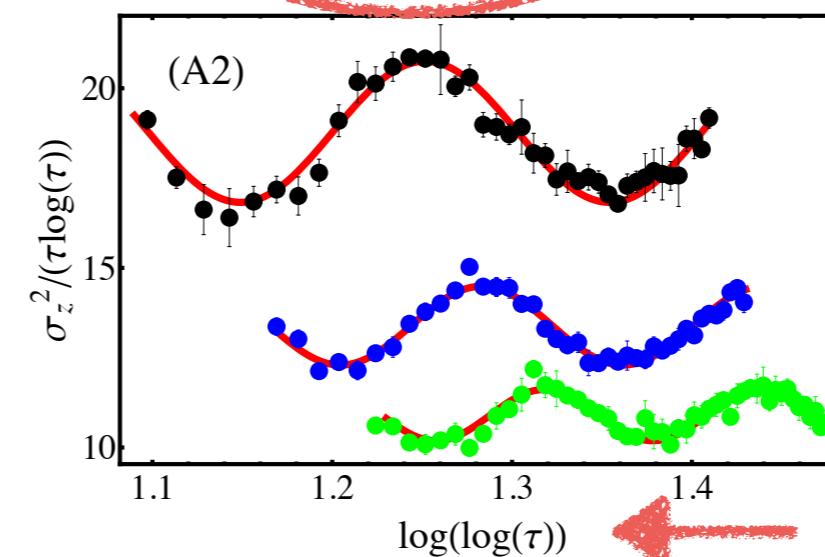
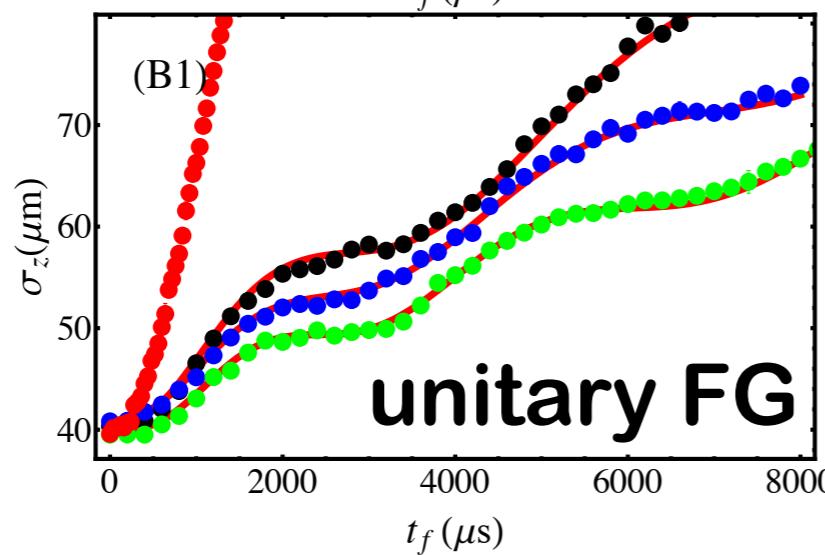
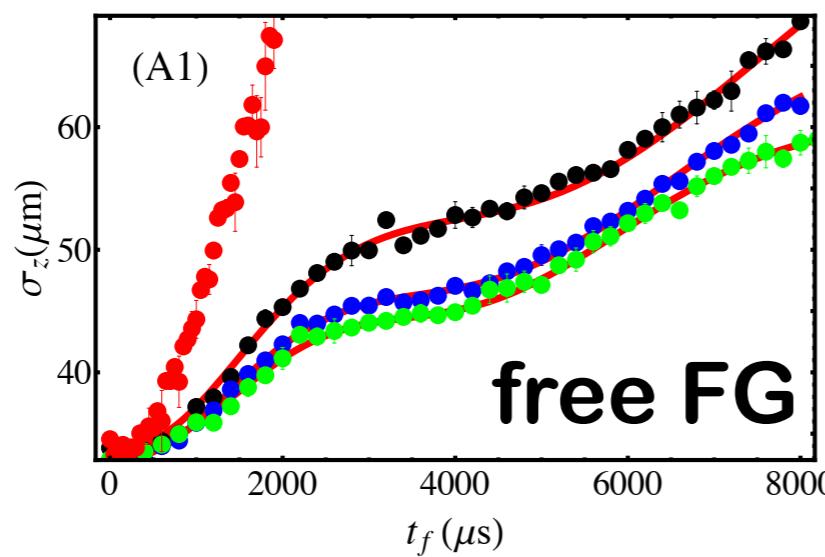
Super-Efimovian expansion

Z.-Y. Shi et al., PRA (2017); S. Deng et al., PRL (2018)



$$\omega = \frac{1}{2t} \sqrt{1 + \frac{g}{\ln^2 t}}$$

$$\langle \vec{x}^2 \rangle \sim t \ln t [A + B \cos(\gamma \ln \ln t + \varphi)]$$



log-log
periodic
oscillation

4. Bulk viscosity

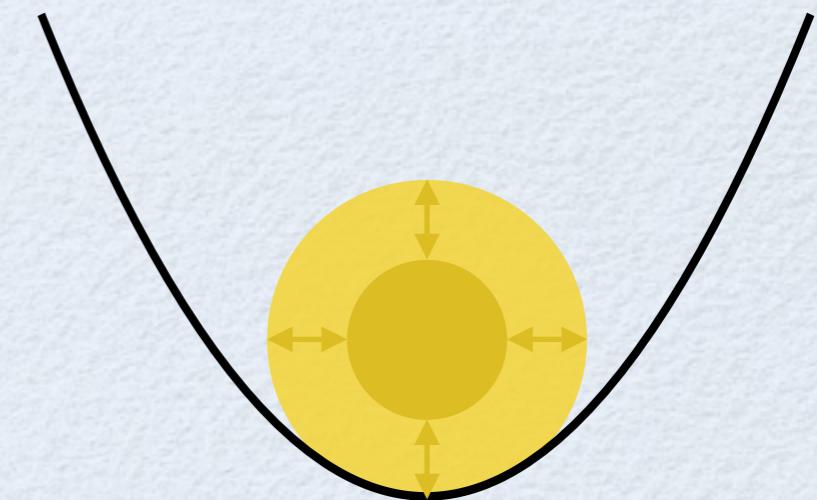
D. T. Son & M. Wingate, Ann Phys (2006)
D. T. Son, PRL (2007)
K. Fujii & Y. Nishida, arXiv:1807.07983

Microscopics @ $a=\text{infinite}$

Undamped “breathing mode”

for any scale invariant systems
confined by harmonic potential

→ **Vanishing bulk viscosity !?**



When coupled with external **gauge field** & **metric**

$$S = \int dt d^d \vec{x} \sqrt{g} \left(i\psi^\dagger \overset{\leftrightarrow}{D}_t \psi - \frac{g^{ij}}{2m} \vec{D}_i \psi^\dagger \vec{D}_j \psi + \mathcal{L}_{\text{int}} \right)$$

is invariant under

- **Gauge transformation** $\psi \rightarrow e^{i\chi(\vec{x}, t)} \psi$
- **General coordinate transformation** $\vec{x} \rightarrow \vec{x}'(\vec{x}, t)$
- **Conformal transformation** $t \rightarrow t'(t)$

Hydrodynamics @ $a=\text{infinite}$

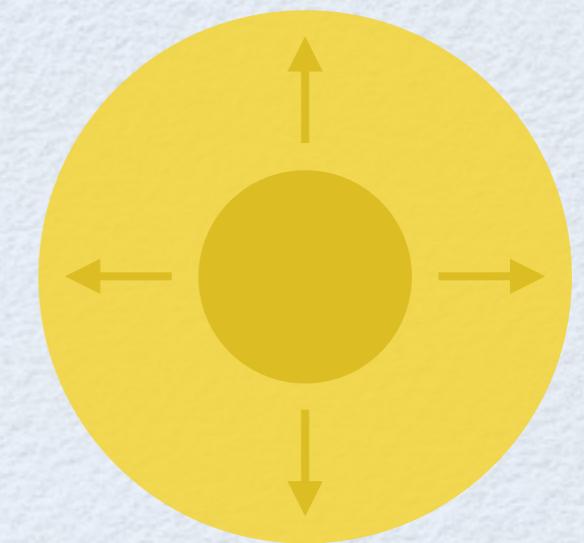
Microscopic symmetries must be inherited by hydrodynamics

Viscous stress tensor **coupled with metric**

$$\pi_{ij} = \zeta \delta_{ij} \partial_k v^k + \text{shear}$$



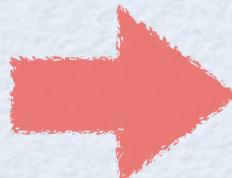
fluid expansion



$$\pi_{ij} = \zeta g_{ij} (\nabla_k v^k + \partial_t \ln \sqrt{g}) + \text{shear}$$

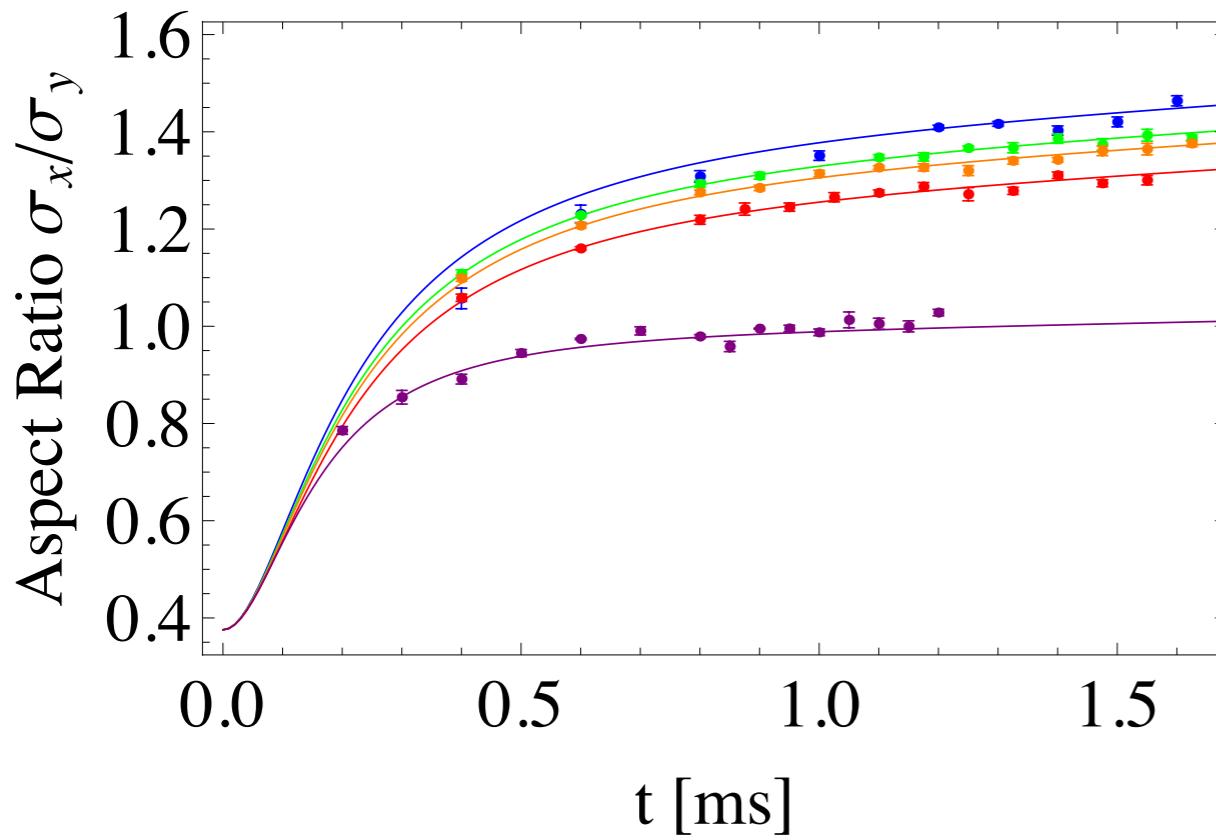
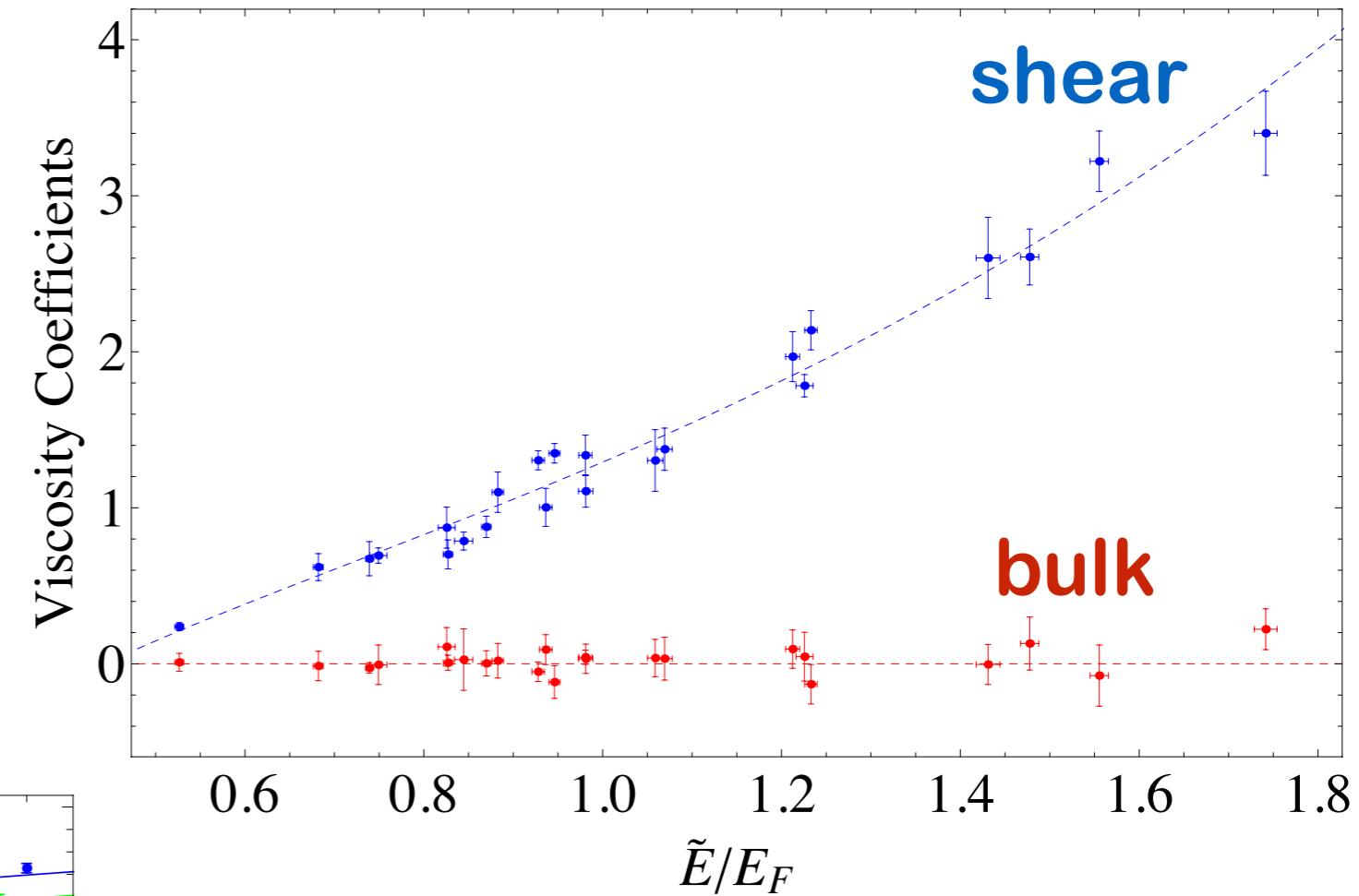
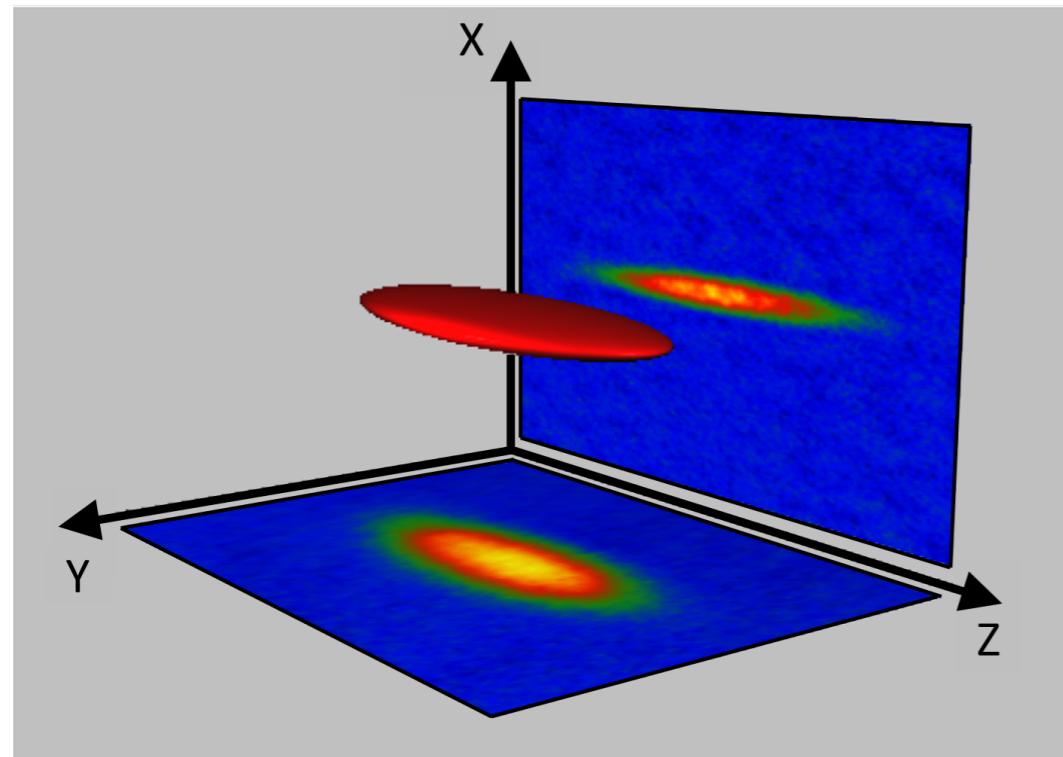
volume expansion

is invariant under general coordinate transformation
but is **NOT** under conformal transformation



Vanishing bulk viscosity @ $a=\text{infinite}$

Hydrodynamics @ $a=\text{infinite}$



$$\frac{1}{N} \int d^3 \vec{x} (\eta, \zeta)$$

$$T/T_F = 0.2 \sim 0.6$$

Microscopics @ $a=\text{finite}$

Scattering length explicitly breaks scale invariance

because $S(a) \rightarrow S(e^s a)$

$$\left(\vec{x} \rightarrow e^{-s} \vec{x}, \quad t \rightarrow e^{-2s} t, \quad \psi \rightarrow e^{(d/2)s} \psi \right)$$

Scale invariance is “formally” **recovered** if

$$a(\vec{x}, t) \rightarrow a'(\vec{x}', t') = e^{-s} a(x, t)$$

spurion field (spacetime-dependent)

Microscopic symmetries must be
inherited by hydrodynamics

$$\pi_{ij}^{\text{bulk}} = \zeta g_{ij} [(\nabla_k v^k + \partial_t \ln \sqrt{g})]$$

is **NOT** invariant under conformal transformation

Microscopics @ $a=\text{finite}$

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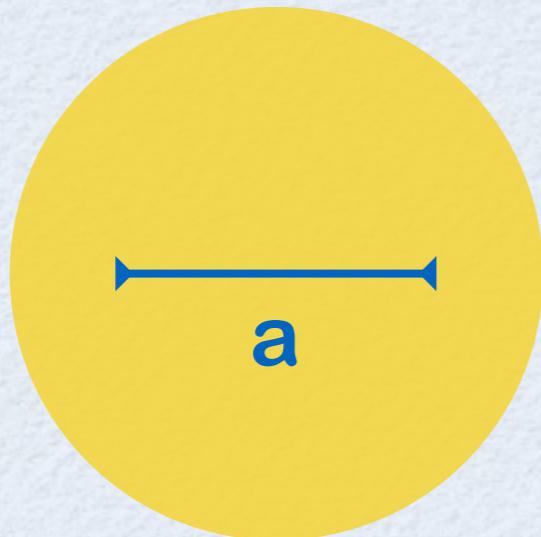
$$\pi_{ij}^{\text{bulk}} = \zeta g_{ij} [(\nabla_k v^k + \partial_t \ln \sqrt{g}) - d (\partial_t \ln a + v^k \partial_k \ln a)]$$

is ~~NOT~~ invariant under conformal transformation

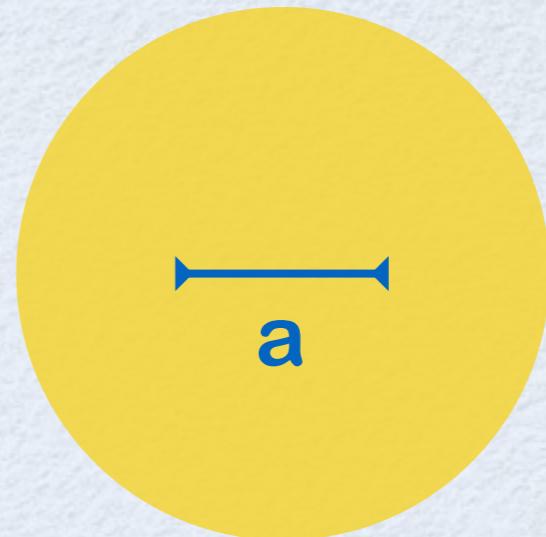
Hydrodynamics @ $a=\text{finite}$

$$\pi_{ij}^{\text{bulk}} = \zeta g_{ij} [(\nabla_k v^k + \partial_t \ln \sqrt{g}) - d (\partial_t \ln a + v^k \partial_k \ln a)]$$

expansion
of fluid



contraction of
scattering length



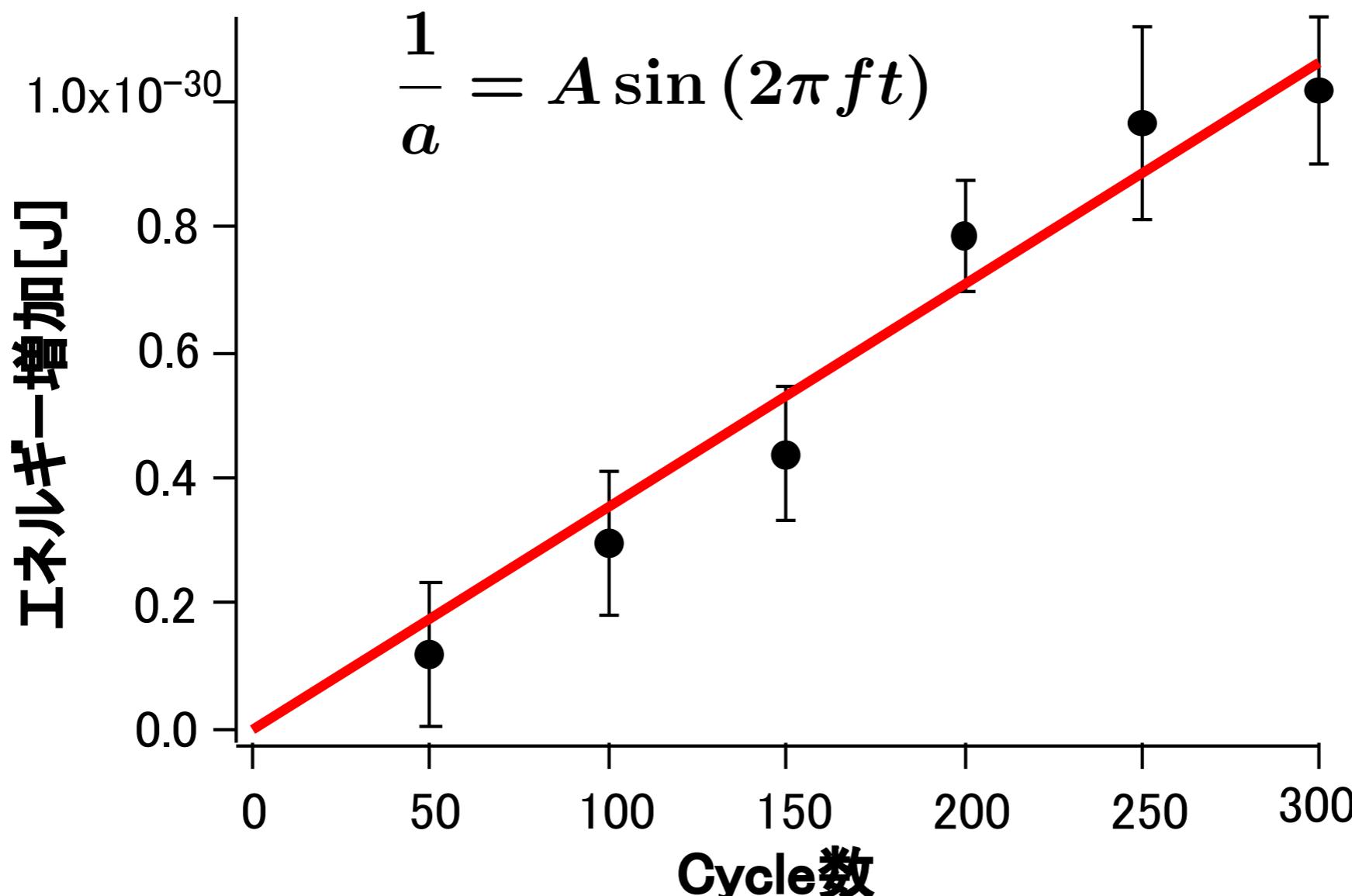
Entropy & energy production in stationary systems



$$T \dot{S} = \frac{d^2 \zeta}{a^2} \dot{a}^2 + O(\dot{a}^3)$$

$$\dot{\mathcal{E}} = \frac{c}{m \Omega_{d-1} a^{d-1}} \dot{a} + \frac{d^2 \zeta}{a^2} \dot{a}^2 + O(\dot{a}^3)$$

Hydrodynamics @ $a=\text{finite}$



Ongoing experiment
toward extraction of bulk viscosity

$$\dot{\varepsilon} = \frac{c}{m\Omega_{d-1}a^{d-1}\dot{a}} + \frac{d^2\zeta}{a^2}\dot{a}^2 + O(\dot{a}^3)$$

Plan Summary of this talk

I would like to “review” have “reviewed”
conformal symmetry in nonrelativistic systems,
its consequences on **(hydro)dynamics**,
and related experiments with **ultracold atoms**.

1. Schrödinger algebra
2. Breathing mode
3. Efimovian expansion
4. Bulk viscosity