Large N, small N, and adiabatic continuity

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Summarizes work by/with many people: D. Dorigoni, G. Basar, E. Poppitz, M. Shifman, M.Unsal, L. Yaffe, …

Big picture

Goal: understand some physically interesting quantity $\mathcal{O}$

This is a resurgence workshop. We think in terms of

$$\mathcal{O}(\lambda) \simeq \sum_n p_n \lambda^n + \sum_c e^{-\frac{S_c}{\lambda}} \sum_k p_{k,c} \lambda^k + \cdots$$

in QM, QFT, string theory…

But what is λ in QFT context?

- Usually λ is a running coupling
- Often it isn't small at the energy scales of interest

How do we accomplish anything, then?

The challenge

Idea: find control parameter C, use it to make $\lambda \ll 1$, compute.

petwave.com

But then what do we learn about the original physics?

- **If** there are no phase transitions as a function of C, then we learn a lot.
- With phase transitions we get a disaster (in practice)!
    - Have to understand full resurgence behavior as a prerequisite for making even qualitative predictions.

Two approaches

- Supersymmetry
  - Studied since ~1980s (in the relevant context)
- Adiabatic Compactification
  - Studied since ~2010s, due to Mithat Unsal and collaborators: Shifman, Yaffe, Poppitz, Dunne, Schafer, Sulejmanpasic, Tanizaki, Misumi, AC,…

SUSY

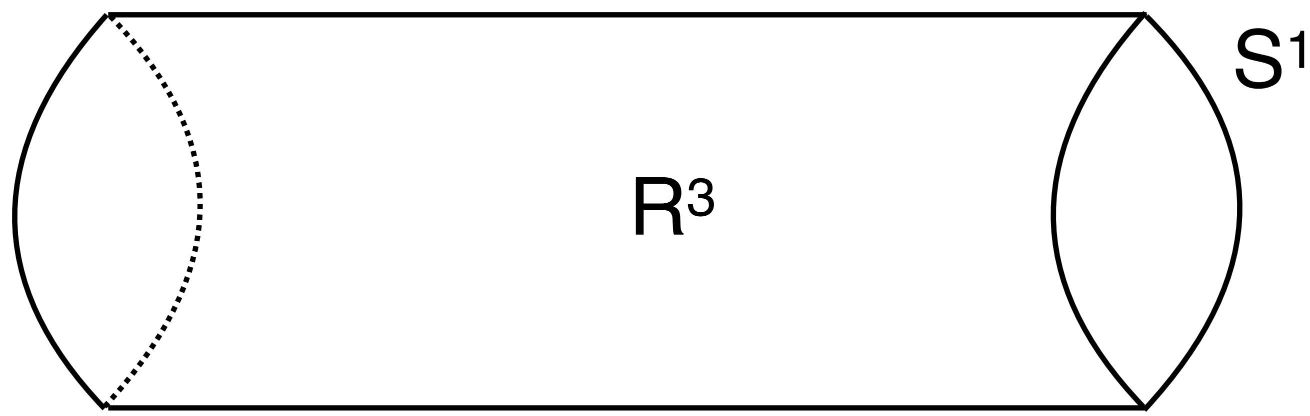
Supersymmetry often naturally gives a control parameter C

- C = < VEV of fundamental scalar field >
  - Asymptotically-freedom ⇒ weak coupling for large C
    - SUSY holomorphy ⇒ results for all C
- Very nice in its own right!
  - Loss of control if SUSY is broken
    - Have to hope there are no phase transitions
  - Whatever you learn might be tied to specifics of SUSY setting: specifics of matter content & interactions…
- Resurgence structure often very different from generic expectations.
  - Cancellations hidden, have to be decoded.

Adiabatic compactification

Unsal, Yaffe, Shifman, … 2008-onward

Idea: break 4D Lorentz, but as little as possible!



If circle size L is small, can get weak coupling by asymptotic freedom

- NB: non-compact $\mathbb{R}^3 \Rightarrow$ symmetries can break spontaneously.
- Large L: some symmetries preserved, others spontaneously broken!
- In practice the small L limit is useful **only** if we get same symmetry breaking pattern at large L and small L
  - Assume symmetry breaking pattern doesn't change at intermediate L - checkable by lattice simulations.

Plan of the talk

Focus will be on adiabatic compactification

1. Fixed N - already done by Mithat, so I'll be brief.
    - Reminder about mass gap and remark on renormalons …
2. 't Hooft large N limit: volume independence, and Hagedorn
3. Small-circle large N limit: emergent extra dimension, and the fate of the mass gap.

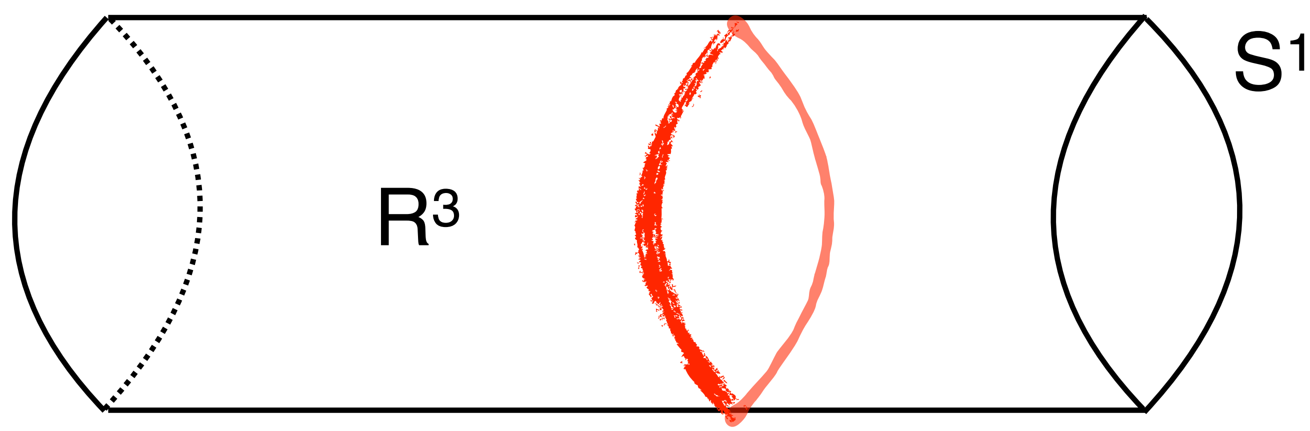
emergent dimension

physical dimension

Part 1

Small L, fixed N.

Self-Higgsing



When gauge theories are compactified on S^1, tr(Polyakov loop) is an observable

$$\Omega = \mathcal{P}e^{i\oint A_3} = \mathrm{diag}\left(e^{i\phi_1}, \ldots, e^{i\phi_N}\right)$$

Eigenvalues are determined dynamically. Their distribution is very important!

Confinement and center symmetry

Heuristically, Polyakov loop associated to confinement

$$\mathrm{tr}\,\Omega^k \sim \exp\left(-\beta F_{\mathrm{k\ quarks}}\right)$$

Confinement ~ infinite cost to have $k \neq N$ excess fundamental quarks.

- N quarks make a baryon, and baryon has finite energy.
- So expect confinement to be associated with

$$\mathrm{tr}\,\Omega^k = 0, \;\; k \neq 0 \bmod N$$

Indeed, YM (without fundamental quarks) has Z_N center symmetry

$$\mathrm{tr}\,\Omega \to \omega\,\mathrm{tr}\,\Omega, \;\; \omega = e^{2\pi i/N}$$

unbroken center $\Rightarrow \langle \mathrm{tr}\,\Omega^k \rangle = 0,\, k \neq 0 \bmod N$

Center symmetry and self-Higgsing

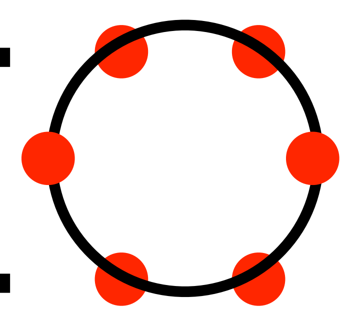
$$\frac{1}{2g^2}\int d^4x\,\mathrm{tr}\,F_{MN}^2 \;\longrightarrow\; \frac{L}{2g^2}\int d^3x\,\mathrm{tr}\,F_{\mu\nu}^2 + (D_\mu A_3)^2$$

If $\langle \mathrm{tr}\,\Omega^k\rangle = 0 \Rightarrow$ "$\langle A_3\rangle \neq 0$". A_3 ~ compact adjoint Higgs field!

Non-coincident eigenvalues for Ω ⇒

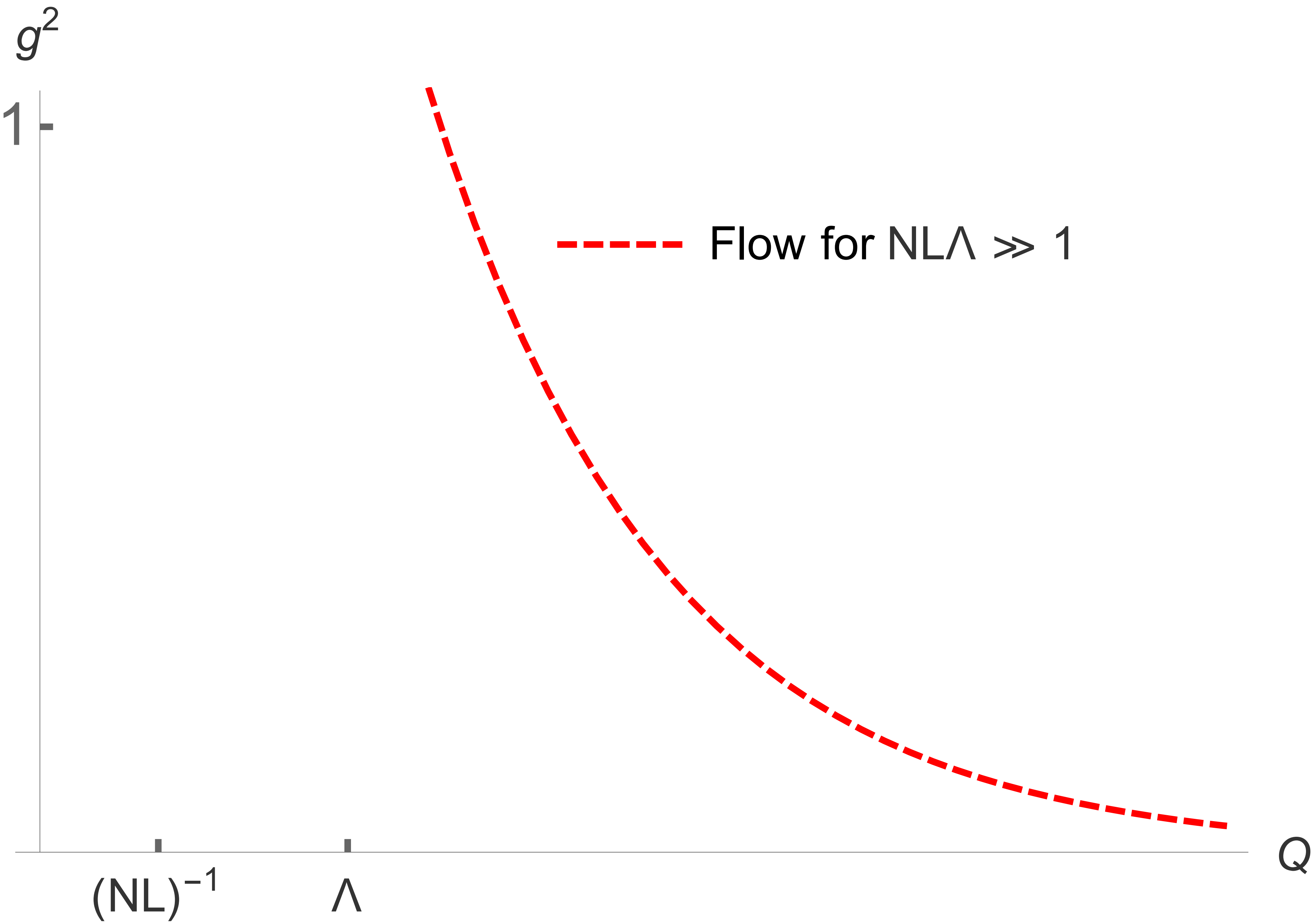
“broken” gauge group SU(N) → U(1)$^{N-1}$ in long-distance 3D EFT

$$\mathrm{tr}\,\Omega^n = 0\ \forall |n| < N \Rightarrow \Omega \sim \mathrm{diag}(1, \omega, \ldots, \omega^{N-1})$$

tr [] = 0

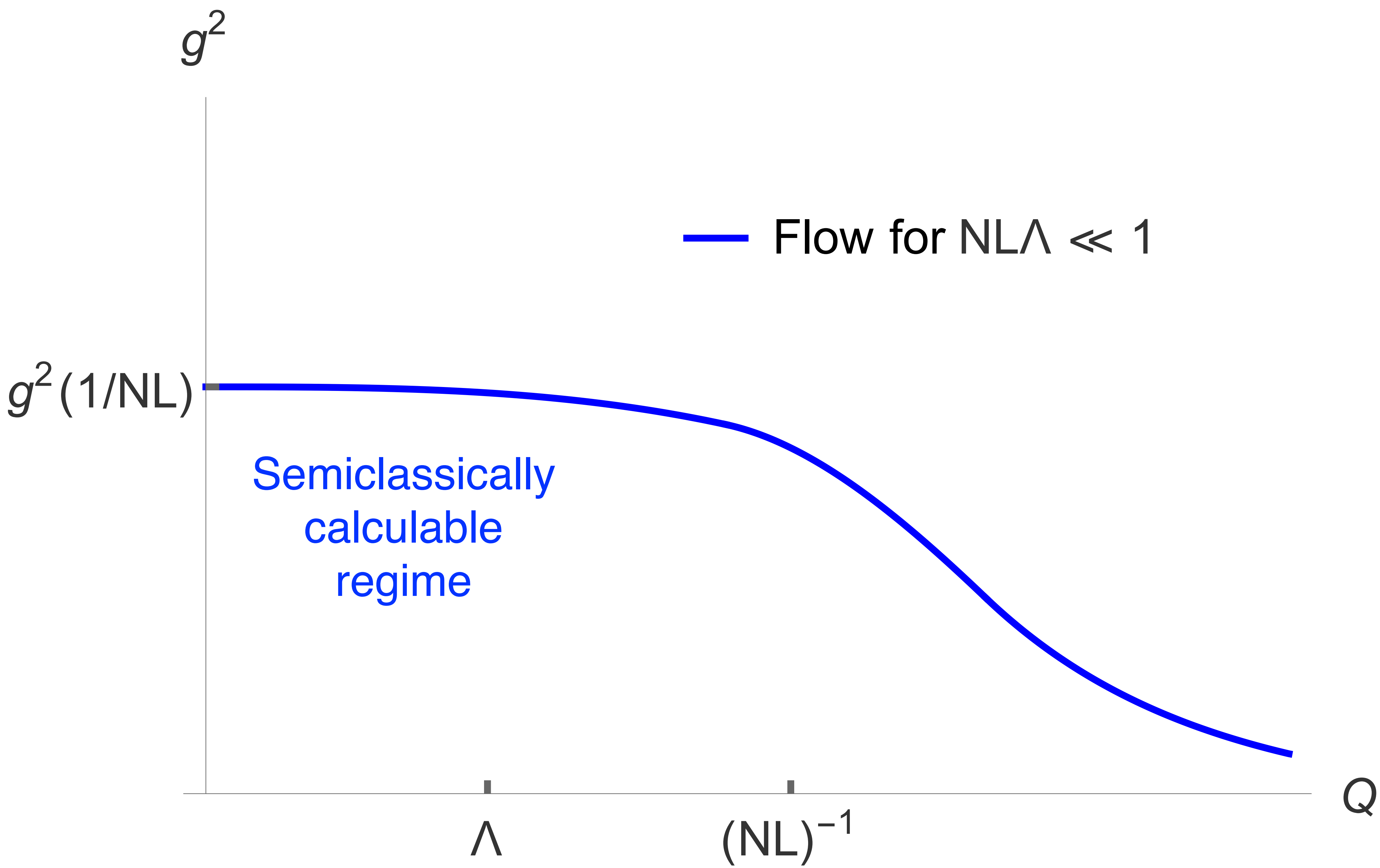
W-boson mass scale: $m_W = 2\pi/NL$

Coupling flows with center symmetry on $R^3 \times S^1$



The $NL\Lambda \gg 1$ regime is strongly-coupled at long distances for all L!

Coupling flows with center symmetry on $R^3 \times S^1$



The $NL\Lambda \ll 1$ regime gives a weakly-coupled theory at all scales!

Preservation of center symmetry

Mithat already explained that preserving center symmetry at small L is hard.

- To the extent L = 1/T, center symmetry “wants” to break!
- This can be avoided using several ingredients:
  - Double-trace deformations
  - Light adjoint fermions with periodic BCs

With you favorite method, you can ensure center symmetry is preserved at small L. Then what?

Small L effective field theory

Suppose N is fixed and $L\Lambda \ll 1$, with center preserved.

- Thanks to adjoint Higgsing, lots of stuff is heavy: $m_W/\Lambda \sim 1/(NL\Lambda) \to \infty$
- Integrate out the manifestly heavy stuff! What remains?
  - $N-1$ Cartan gluons, from 3D components of gluon field strength
  - Working out their fate is crucial!

Small L limit in perturbation theory

N - 1 Cartan gluons are classically gapless.

$$F^i_{\mu\nu} = g^2/(2\pi L)\epsilon_{\mu\nu\alpha}\partial^\alpha\sigma^i$$

$$S_\sigma = \int d^3x \, \frac{g^2}{8\pi^2 L}(\partial_\mu\vec{\sigma})^2.$$

- σ^i shift symmetry $\Longleftrightarrow$ conservation of magnetic charge.
- But there are no magnetic monopoles in perturbation theory.
- So σ^i are massless to all orders in perturbation theory.

Finite-action field configurations

Since $SU(N) \to U(1)^{N-1}$, 4D BPST instanton breaks up into N 'monopole-instantons' with action $S_I/N = 8\pi^2/\lambda$

$N-1$ have $Q = 1/N$, magnetic charges ± 1 under nearest-neighbor U(1)'s. The N^{th} one is 'Kaluza-Klein' monopole.

Lee, Yi; Kraan, van Baal; 1998

't Hooft amplitude

$$\mathcal{M}_i \sim e^{-8\pi^2/\lambda} e^{i(\sigma_i - \sigma_{i+1})} \qquad \lambda = g_{\mathrm{YM}}^2 N$$

$$\text{core size} \sim NL$$

$$\text{typical separation} \sim NLe^{+8\pi^2/(3\lambda)}$$

$\lambda \ll 1$ when $NL\Lambda \ll 1$, so dilute gas approximation is justified.

- Contrast with usual IR disasters with instantons in YM!

Weak coupling confinement

Unsal,
Yaffe, Shifman, Poppitz,
Sulejmanpasic,
…

$$V(\sigma) \sim -m_W^3 e^{-8\pi^2/\lambda} \sum_{i=1}^{N} \cos(\sigma_i - \sigma_{i+1}) + \cdots$$

Dual photons get a mass gap:

$m_p \sim m_W e^{-4\pi^2/\lambda} |\sin(\pi p/N)|$, p = 1, … , N - 1.

Concrete realization of old Mandelstam, ’t Hooft, Polyakov dreams: mass gap driven by proliferation of magnetic monopoles.

String tension also calculable, and is finite. Behaves just as expected from YM.

Poppitz, Erfan S. T, Anber, … 2017 onward

Can also profitably study θ dependence.

Unsal, Yaffe, Tanizaki, Misumi, Fukushima, AC, Poppitz, Schafer, …

Resurgent ambiguities in adiabatic compactification

Beyond leading order in semi-classical expansion, neutral bion amplitudes are (usually) ambiguous:

$$\left[\mathcal{M}_i \overline{M}_i\right] \sim \pm i\, e^{-16\pi^2/\lambda}$$

(exactly massless adjoint fermions change the story)

This ambiguity does not vanish exponentially with N.

Arises from quasi-zero mode integration. A kind of generalized instanton effect, so it should be possible to relate to some proliferation of Feynman diagrams.

- Feynman diagrams on $\mathbb{R}^3 \times S^1 \neq$ Feynman diagrams on $\mathbb{R}^4$
- Color-sums related to S^1 momentum sums

Eguchi, Kawai; Gross, Kitazawa, …

Comments on renormalons

What is a renormalon?

My preferred definition: it is an ambiguity in the Borel resummation of perturbation theory, with a size which doesn't vanish at large N.

- Other definitions are used in some of the literature. I think this one is better, for reasons I'll explain next.

What is the interplay of renormalons with adiabatic compactification?

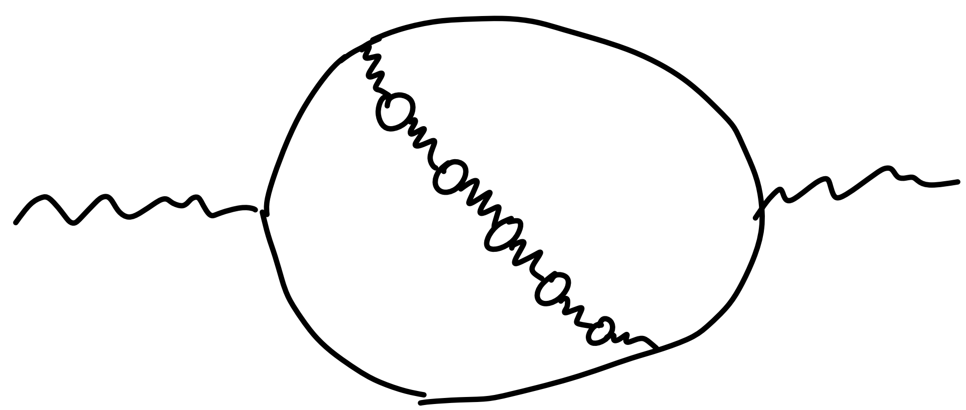
4D: Argyres, Unsal; Anber, Sulejmanpasic; Ashie, Ishikawa, Takaura, **Morikawa**, Suzuki, Takeuchi, …

2D: Dunne, Unsal; Fujimori, Kamata, Misumi, Nitta, Sakai;

…

Comments on renormalons

On $\mathbb{R}^4$, renormalons come from diagrams like this:



Renormalons arise from an IR divergence in these diagrams, give rise to ambiguities in Borel summed perturbation theory, so in YM

$$\text{ambiguity} \;\sim \pm\, i e^{-\#/\lambda}$$

is such that it can be cancelled by an ambiguity in some ‘condensate’, e.g. $\langle \mathrm{tr} F_{\mu\nu}^2 \rangle \sim \Lambda^4$. Remember: $\Lambda \sim \mu e^{-8\pi^2/(\lambda \cdot 11/3)}$

- Has to be like this for consistency! Λ is the only scale.

Comments on renormalons

What should we expect with adiabatic compactification?

Adiabatic compactification **eliminates IR divergences** by design!

- Are renormalons gone? The Feyman diagrams that gave them on $\mathbb{R}^4$ aren't divergent any more.
- If we *define* renormalon = certain Feynman diagrams with certain divergences, then, yes they're gone.
  - My view: not a good definition. Number and value of individual Feynman diagrams is not a physical invariant!

Comments on renormalons

Distinction between Borel singularities "from IR divergences" or "from number of individual diagrams" is not physical.

- What matters: **size of effect**, how to understand it, how it fits with other dynamics, and so on.

Hence my preferred definition: renormalons are an ambiguity in the Borel resummation of perturbation theory, with a size which doesn't vanish at large N.

- This is the definition assumed in the Argyres-Unsal and Dunne-Unsal papers that kicked off modern QFT resurgence.

So what should we expect about renormalons at small L?

Renormalons in adiabatic compactification

With adiabatic compactification, there is another scale in addition to Λ, namely $1/L$. The physics depends on both!

- Renormalons should depend on 1/L as well, and indeed they do:

$$\text{ambiguity} \;\sim \pm\, i e^{-\#/\lambda} \sim \Lambda^p (NL\Lambda)^k$$

see e.g. Dunne, Shifman, Unsal 2015

More precisely: we know non-perturbative quantities have ambiguities like this, due to “neutral bions”.

- It’s harder to explicitly match it to perturbation theory…

but see talk by O. Morikawa on Thursday (Wed in US)!

In any case, this perspective implies that location of Borel singularities must flow as a function of $NL\Lambda$.

Part 2

’t Hooft large N limit

Large N vs adiabatic compactification

Suppose $N \to \infty$ with λ and all other parameters fixed in a theory with adiabatic circle-size dependence.

- What does it imply?

- Dependence on L is very adiabatic: no dependence at all
  - "Eguchi-Kawai reduction"/"large N volume independence"
  - in the adjoint QCD example: spectacular Bose-Fermi cancellations
    - More subtle (and not well understood) with double-trace deformations

Hagedorn instability

Put any confining large N theory on R^3 x S^1_β

$$Z(\beta) = \mathrm{Tr}\; e^{-\beta H} = \int dE\, \rho(E) e^{-\beta E}$$

$$\rho(E) \to e^{+\beta_H E},\;\; \beta_H \sim \Lambda_{\mathrm{QCD}}^{-1}$$

Once $\beta < \beta_H$, energy integral diverges! No big change if $\mathbb{R}^3$ replaced by some compact manifold M.

$\Rightarrow$ phase transition at or below T_H to a phase where $\rho(E)$ scales differently.

This is the deconfinement transition to the quark-gluon plasma phase!

Confinement versus Hagedorn

Can deconfinement as a function of β be avoided?

$$\tilde{Z}(L) = \mathrm{Tr}\ (-1)^F e^{-LH}$$
$$= \int dE\ [\rho_B(E) - \rho_F(E)]\, e^{-LE}$$

'all' we need are precise-enough cancellations between $\rho_B(E)$ and $\rho_F(E)$.

With SUSY $\rho_B(E) = \rho_F(E), E > 0$, so no problem.

What about *without* SUSY?

Required cancellations

Expect Hagedorn scaling for both ρ_B and ρ_F. More precisely:

$$\rho_B(E) \to e^{+\beta_1 E} \sum_n p_{n,1} E^{-n} + e^{+\beta_2 E} \sum_n p_{n,2} E^{-n} + \cdots$$
$$+ \sum_n \tilde{p}_{n,B} E^{-n} + \cdots$$

$$\rho_F(E) \to e^{+\beta_1 E} \sum_n p_{n,1} E^{-n} + e^{+\beta_2 E} \sum_n p_{n,2} E^{-n} + \cdots$$
$$+ \sum_n \tilde{p}_{n,F} E^{-n} + \cdots$$

All terms with positive exponentials must be identical to avoid an instability!

Avoiding Hagedorn

This degree of conspiracy between bosons and fermions seem fanciful without supersymmetry.

And indeed, it doesn't work in large N QCD.

Fermionic states — baryons —only for odd N, which are heavy at large N.

$$\tilde{Z}_{\mathrm{QCD}}(\beta) = \mathrm{tr}\,(-1)^F e^{\beta H} = Z_{\mathrm{QCD}}(\beta) = \mathrm{tr}\, e^{\beta H}$$

Have to look further afield for a working example.

A special non-SUSY QFT

Consider SU(N) YM coupled to $1 \leq N_F \leq 5$ flavors of massless adjoint Majorana fermions: **adjoint QCD**

- Confining on $\mathbb{R}^4$ if N_F is not too close to 5.

$$N_F = 1 \Rightarrow \mathcal{N} = 1 \text{ super YM}$$

Otherwise, no SUSY.

Adjoint QCD

Adjoint QCD has lots of light fermionic states for any N_F

$$\mathrm{tr}\,(F_{\mu\nu}^2\lambda_a)\,|\,0\rangle, \mathrm{tr}\,(F_{\mu\nu}^2\lambda_a\lambda_b\lambda_c)\,|\,0\rangle, \ldots$$

But no SUSY, because there are $2(N^2-1)$ microscopic bosons and $2N_F(N^2-1)$ microscopic fermions

Punchline 1

No phase transitions with $\tilde{Z}(\beta) = \mathrm{tr}\,(-1)^F e^{\beta H}$ even when $1 < N_F \leq 5$.

- Originally proposed by Kovtun, Unsal, Yaffe, 2007.
- Later analytic and many numerical lattice analyses basically proved it.

At large N, KUY observation has a striking implication:
all Hagedorn instabilities cancel, without SUSY.

Basar, AC, Dorigoni, Unsal,
2013

Punchline 2

AC, Shifman, Unsal 2018

What could be left after the Hagedorn cancellations?

Naive guess: a few particles worth of 4d degrees of freedom.

$$\tilde{\rho}(E) \sim \exp(V^{1/4}E^{3/4}) \;\Leftrightarrow\; \log\tilde{Z} \sim \beta^{-3}V$$

Claim: actually get at most a 2d density of states

$$\tilde{\rho}(E) \sim \exp(\sqrt{\ell E}) \;\Leftrightarrow\; \log\tilde{Z} \sim \ell/\beta$$

$$\ell \sim \int_M d^3x\,\sqrt{g}\,\mathcal{R}$$

Just like in SUSY QFT, despite lack of SUSY!

Punchline 2

More precisely, in SU(N) x U(1) ~ U(N) adjoint QCD in the large N limit we get

$$\log Z = c_3 \beta^{-3} \int_M d^3x \sqrt{g} + c_1 \beta^{-1} \int_M d^3x \sqrt{g}\, \mathcal{R} + \cdots$$

$$c_3 \sim \frac{1}{N^2}$$

If we take N large with geometry fixed,

$$\log \tilde{Z}_{\text{adjoint QCD}} = c_1 \beta^{-1} \int d^3x \sqrt{g}\, \mathcal{R} + \text{non-singular}$$

No sharp cancellations at finite N. But at large N, cancellations are as strong as in SUSY theories.

Origin of cancellations

First, how do we know whether or not center is broken when L is small?

- One-loop calculations giving

$$V(\Omega) \sim \frac{(N_F - 1)}{L^4} \sum_{n=1}^{\infty} \frac{1}{n^4} |\mathrm{tr}\, \Omega^n|^2$$

What is the condition for results like this to be valid?

Center symmetry in YM theory

Polyakov loop size $\sim \beta$. When $\beta\Lambda \ll 1$, quantum corrections are small, can compare center-broken and center symmetric extrema.

- If this was false, we wouldn't even know that hot YM is deconfined!

More careful argument: IR divergences, magnetic and electric screening scales, turning on non-trivial holonomy reduces IR divergences.

So for pure YM theory with $L\Lambda \ll 1$, one gets

$$V(\Omega) \sim -\frac{1}{L^4}\sum_{n=1}^{\infty}\frac{1}{n^4}|\mathrm{tr}\,\Omega^n|^2 \sim -\frac{1}{L^4}\sum_{n=1}^{\infty}\frac{1}{n^4}|\mathrm{tr}\,\mathbf{1}_{\mathbf{N}}|^2 \sim \frac{-N^2}{L^4}$$

This implies $\log Z \sim N^2\beta^{-3}$ · (spatial volume)

Center symmetry in adjoint QCD

Things change in adjoint QCD with periodic boundary conditions

One-loop calculation:

$$V_{\text{eff}}(\Omega) = \frac{2(n_F - 1)}{\pi^2 \beta^4} \sum_{n \geq 1} \frac{1}{n^4} \left| \operatorname{tr} \Omega^n \right|^2 ,$$

Potential flips! Minima turn into maxima, and vice versa.

$\mathbb{Z}_N$ center symmetry preserved on new minimum

$$\Omega \sim \operatorname{diag}(1, e^{2\pi i/N}, \cdots , e^{2\pi i(N-1)/N})$$

$$\langle \operatorname{tr} \Omega^n \rangle = 0, \; n \neq 0 \bmod N$$

Center symmetry in adjoint QCD

Evaluate effective potential on its minimum to get $\log \tilde{Z}$

$$\log \tilde{Z} \sim \frac{1}{N^2}\beta^{-3}\mathrm{Vol}_M$$

Physically, results from +/- grading for physical states.

What happened to the individual quarks and gluons?

Quarks and gluons contribute with $\mathbb{Z}_N$ phases
due to Ω. Huge cancellations!

Only with a trivial — center-breaking — Polyakov loop do all fields contribute with one sign!

Center symmetry in adjoint QCD

To see how it works pick a gauge where A_4 = const.

$$\Omega \sim (e^{i\phi_1}, \ldots, e^{i\phi_N})$$

Holonomy ~ imaginary chemical potential for colored fields

$$V_{\text{eff}}(\Omega) = \frac{2(n_F - 1)}{L} \sum_{a,b=1}^{N} \int \frac{d^3p}{(2\pi)^2} \log\left(1 - e^{-pL} e^{i\phi_a - i\phi_b}\right)$$

Implies that contributions weighed by phases in general.

When $\boldsymbol{\Omega} \sim \mathbf{1}_N$, no phases $\Rightarrow N^2$ scaling

But with center symmetry, massive cancelations that convert $N^2 \to 1/N^2$

Center symmetry in adjoint QCD

Do cancelations work beyond one loop? Yes.

$$u_n \equiv \frac{1}{N} \langle \mathrm{tr}\, \Omega^n \rangle$$

AC, Shifman, Unsal 2018

$$V_{\mathrm{eff}} = \beta^{-4} \left(N^2 f_0(\{u_n\}) + N^0 f_1(\{u_n\}) + N^{-2} f_2(\{u_n\}) + \ldots \right)$$

Write f_g as sums of terms with different number of color traces summed over windings.

- Finiteness of V_{eff} requires certain properties of these sums
- Some special terms can be estimated explicitly.
- Combination of these features implies all—order result:

$$\log \tilde{Z}_{\text{adjoint QCD}} = c_1 \beta^{-1} \int d^3x \sqrt{g}\, \mathcal{R} + \text{non-singular}$$

Beyond perturbation theory

Discussion so far has been perturbative.

- Non-perturbative effects are irrelevant!
  - Non-perturbative effects weighed by powers of $e^{-1/\lambda}$
  - Dimensional transmutation turns this into powers of the strong scale Λ
- Dimensional analysis $\Rightarrow$ non-perturbative physics can't affect coefficient of β^{-3} in $\log \tilde{Z}$

Perturbation theory is all that matters!

Summary of part 2

If we take a standard large N limit in theories with adiabatic continuity, fun and magical things happen:

- Volume dependence disappears entirely
- In best-understood example, adjoint QCD, we find amazing SUSY-like spectral cancellations.
  - Would be nice to understand in detail how twisted Eguchi-Kawai and double-trace deformed YM manage to avoid deconfinement from this perspective…

Of course, there is also something less fun:

- We can’t calculate almost anything at large N!
- Field-theory semiclassical methods seem useless.
  - Is there any easy way around that?

Part 3

Small-circle large N limit

Paths to large N

’t Hooft limit: fix $\lambda = g^2 N$ at the UV cutoff as well as all physical parameters, and take $N \to \infty$

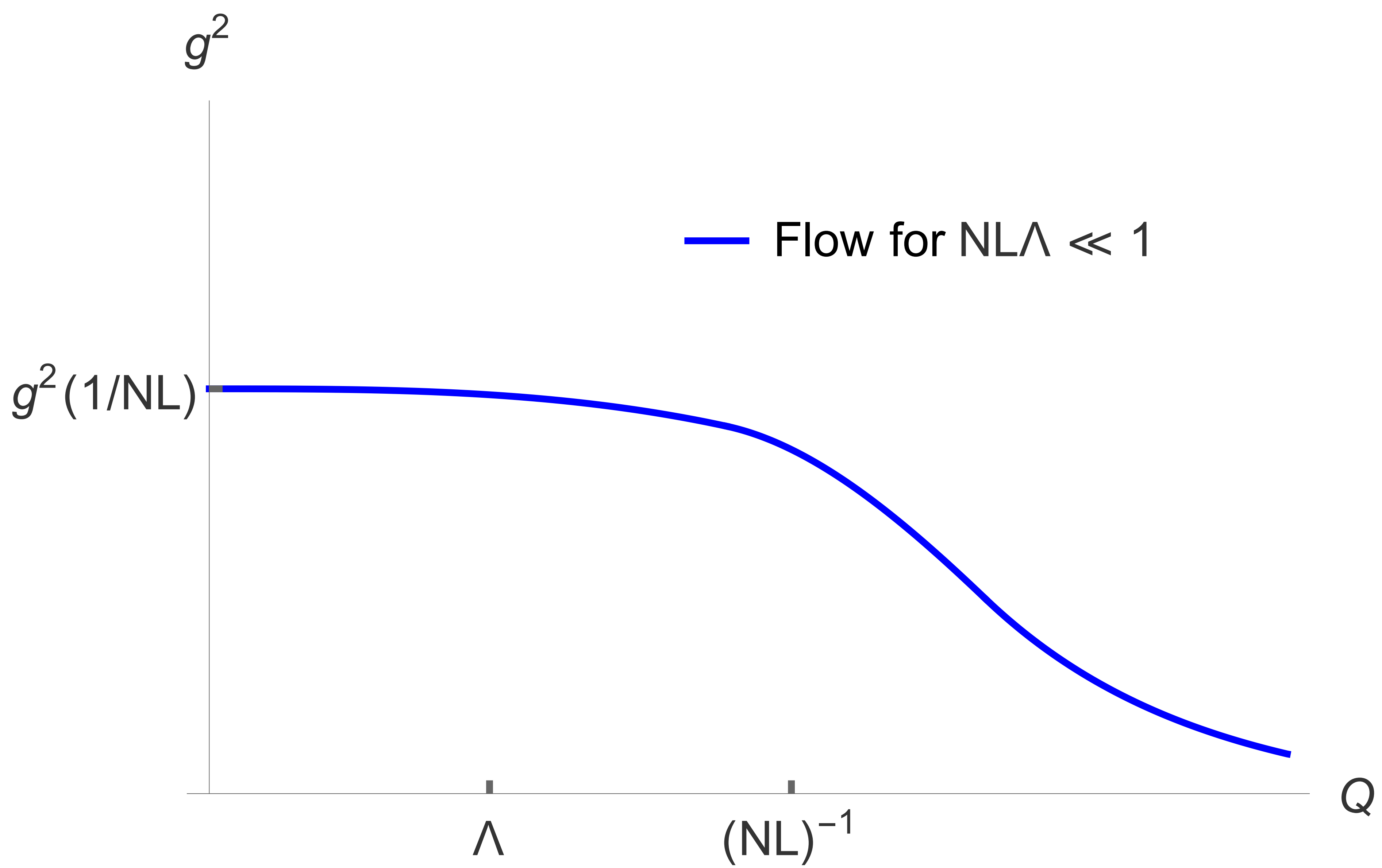
We can’t solve most interesting 4d gauge theories in this limit.

- But we do have adiabatic compactification that lets us solve some of them on small circles at finite N
- If we take an ’t Hooft large N limit, we lose control.
  - Large N volume independence forces this!
- Is there some other large N limit where we can keep computational control?

Small circle large N limit

To keep control, we have to ensure $m_W \gg \Lambda$.

$$NL\Lambda \ll 1$$



Small circle large N limit

To keep control, we have to ensure $m_W \gg \Lambda$.

$$\boxed{NL\Lambda \ll 1}$$

New large N limit: fix $\eta = NL\Lambda, \lambda_{UV} = g^2 N$ and take $N \to \infty$.

- If $\eta \gg 1$ we have no control - but it should look volume independent.
- If $\eta \ll 1$, we have complete control. Can calculate everything: mass gap, symmetry breaking, renormalons, theta dependence, …
- Solvable large N limits are extremely rare, so we should take them very seriously.

Large N super-YM at small $\eta = NL\Lambda$

I'll explain how things go for 4d $\mathcal{N} = 1$ pure SU(N) super-YM theory

- No phase transitions for any L.
  - Confines and spontaneously breaks discrete chiral symmetry at any L.
- Weakly coupled when $\eta \ll 1$, even at large N.

What's the low energy spectrum at large N?

Light fields at small L

For long distances $\ell \gg m_W = \Lambda\eta^{-1}$, dynamics is Abelian:

$$SU(N) \to U(1)^{N-1}$$

Without SUSY, only classically massless fields are the N - 1 "Cartan gluons".

NB: they're physical!

$$F^j_{\mu\nu} = \frac{1}{N}\sum_{p=0}^{N-1} e^{2\pi i j p/N}\,\mathrm{tr}\left(\Omega^p F_{\mu\nu}\right)$$

(added fictitious j = 0 mode for notational simplicity; decouples exactly.)

'color' label j = Fourier transform of winding label p.

Light fields at small L

For long distances $\ell \gg m_W = \Lambda \eta^{-1}$, dynamics is Abelian:

$$SU(N) \rightarrow U(1)^{N-1}$$

With SUSY, classically massless fields in 3D EFT are:

- N-1 Cartan gluons, from 3D components of gluon field strength
- + N-1 ϕ_a scalar fields (A_3 fluctuations)
- + N-1 ψ_a fermion fields (Cartan gluinos)
  - All these fields sit in the same supermultiplet

I'll focus on the Cartan gluons - the other fields come along for the ride.

What is the effective action?

Small L limit in perturbation theory

The Cartan gluons are classically gapless.

$$F^i_{\mu\nu} = g^2/(2\pi L)\epsilon_{\mu\nu\alpha}\partial^\alpha\sigma^i$$

$$S_\sigma = \int d^3x\, \lambda m_W (\partial_\mu\vec{\sigma})^2$$

σ^i shift symmetry $\Longleftrightarrow$ conservation of magnetic charge.

$$J_\mu = \partial_\mu\sigma^i \sim \epsilon_{\mu\nu\alpha}F^{i,\,\nu\alpha}$$

No magnetic monopoles in perturbation theory
$\Rightarrow$ σ^i are massless to all orders in loop expansion

Non-perturbative mass gap

Since $SU(N) \to U(1)^{N-1}$, 4D BPST instanton breaks up into N 'monopole-instantons' with action $S_I/N = 8\pi^2/\lambda$

Monopole-instantons have Q = 1/N,
and magnetic charges +1,-1 under nearest-neighbor U(1)'s

't Hooft amplitude $$\mathcal{M}_i \sim e^{-8\pi^2/\lambda} e^{i(\sigma_i - \sigma_{i+1})} \qquad \lambda = g^2_{\mathrm{YM}} N$$

$$\text{core size} \sim NL$$

$$\text{typical separation} \sim NLe^{+8\pi^2/(3\lambda)}$$

λ « 1 when NLΛ « 1, dilute gas approximation justified at small L.

Contrast with usual IR disasters with instantons in YM!

Monopole-instantons and related excitations induce mass gap.

Long-distance EFT for $\mathcal{N} = 1$ SYM

Since $SU(N) \to U(1)^{N-1}$, 4D BPST instanton breaks up into N 'monopole-instantons' with action $S_I/N = 8\pi^2/\lambda$

super-YM has massless adjoint fermions ⇒ 2 fermion zero modes on monopoles;

$$\mathcal{M}_i \sim e^{i(\sigma_i - \sigma_{i+1})} \lambda_i \lambda_{i+1}$$

Monopole-instantons give interactions for light fermions

Contribute to fermionic potential, not the bosonic one.

Long-distance EFT for $\mathcal{N} = 1$ SYM

Any field configuration with fermion zero modes can't contribute to bosonic potential.

- Since every BPS configuration has fermion zero modes, can't get bosonic potential from **any** BPS objects.
- Mass gap is induced by topologically-trivial solutions with finite action and magnetic charge: "magnetic bions"

Unsal 2007

$$V(\vec{\sigma})\Big|_{\mathrm{super-YM}} \sim -m_W^3 e^{-16\pi^2/\lambda} \sum_{i=1}^{N} \cos(2\sigma_i - \sigma_{i-1} + \sigma_{i+1})$$

Making sense of this is fun challenge for SUSY connoisseurs.

$$m_\sigma\Big|_{\mathrm{super-YM}} \sim m_W e^{-8\pi^2/\lambda}$$

Emergent dimension

More careful look at long-distance EFT for $\mathcal{N}$ = 1 sYM:

$$S_\sigma = \int d^3x\, \lambda m_W (\partial_\mu \vec{\sigma})^2 - m_W^3 e^{-16\pi^2/\lambda} \sum_i \cos(2\sigma_i - \sigma_{i+1} - \sigma_{i-1}) + \cdots$$

N minima due to discrete chiral symmetry breaking $\mathbb{Z}_{2N} \to \mathbb{Z}_2$

- Canonically normalize, expand around any given vacuum:

$$S_\sigma = \int d^3x \sum_{i=1}^{N} \left\{ (\partial_\mu \tilde{\sigma}_i)^2 + M^2 (2\tilde{\sigma}_i - \tilde{\sigma}_{i-1} - \tilde{\sigma}_{i+1})^2 \right\}$$

$$M \sim m_W e^{-8\pi^2/\lambda}$$

$$\Lambda^3 = \mu^3 e^{-8\pi^2/\lambda} \implies M \sim \Lambda \eta^2,\ \eta = NL\Lambda$$

Complete EFT for $\mathcal{N} = 1$ SYM

Package scalars as $\Phi_j = \phi_j + i\sigma_j$, diagonalize quadratic action:

$$S_{\mathrm{EFT}} \sim \int d^3x \sum_{p=1}^{N} \left\{ |\partial_\mu \Phi_p|^2 + M_p^2 |\Phi_p|^2 + \bar{\Psi}_p \partial\!\!\!/ \Psi_p + \frac{M_p}{2} \left(\Psi_{N-p} \Psi_p + \mathrm{h.c.} \right) \right\}$$

$$M_p \sim m_W e^{-8\pi/\Lambda} \sin^2\left(\frac{\pi p}{N}\right) \sim m_W e^{-8\pi^2/\lambda} \frac{p^2}{N^2}$$

Mass gap vanishes at large N!

What's going on?

Emergent dimension

$$S_\sigma = \int d^3x \sum_{i=1}^{N} \left\{ (\partial_\mu \tilde{\sigma}_i)^2 + M^2 (2\tilde{\sigma}_i - \tilde{\sigma}_{i-1} - \tilde{\sigma}_{i+1})^2 \right\}$$

$$M \sim m_W e^{-8\pi^2/\lambda} = \Lambda \eta^2 =: \frac{1}{a}$$

'color' label i = position, difference operators = derivatives

Large circular extra dimension $\tilde{L} = Na$

$$\tilde{L}/L \sim N^2 \eta^{-3}, \qquad \tilde{L}\Lambda = N\eta^{-2}$$

Disappearing mass gap $\Longleftrightarrow$ decompactification of gapless fields

Emergent dimension

On $O(N^0)$ distance scales $\tilde{L} \gg \ell \gg a$

$$S_{\rm YM} \sim \int d^3x \, dy \{ |\partial_\mu \Phi|^2 + |\partial_y^2 \Phi|^2 + \bar{\Psi} \partial\!\!\!/ \Psi + \tfrac{1}{2} (\Psi \partial_y^2 \Psi + \text{h.c.}) \}.$$

spatial Lifshitz scale invariance with z = 2!

$$\vec{x} \to \Omega \, \vec{x}, \ y \to \Omega^{1/z} \, y$$

Summary

Took 4D $\mathcal{N}$ = 1 super-YM, turned on "relevant deformation" — the circle.

- At least if S^1 is small, resulting flow is to a non-trivial IR fixed point!
- Is this some weird SUSY thing?

Emergent dimension far from SUSY point

$$S\big|_{\mathrm{YM}} = \int d^3x\, \lambda m_W (\partial_\mu \vec{\sigma})^2 - m_W^3 e^{-8\pi^2/\lambda} \sum_i \cos(\sigma_i - \sigma_{i+1}) + \cdots$$

$$S\big|_{\mathrm{YM}} = \int d^3x\, \sum_{i=1}^{N} \left\{ (\partial_\mu \tilde{\sigma}_i)^2 + M^2 (\tilde{\sigma}_i - \tilde{\sigma}_{i+1})^2 \right\}$$

Fourier-space "mass" ~ $M\,|\sin(\pi p/N)\,|$

$$S\big|_{\mathrm{YM}} \sim \int d^3x\, dy \{ |\partial_\mu \sigma|^2 + |\partial_y \sigma|^2 \}$$

Long distance theory again scale invariant, but now with z = 1

- Tuning to SUSY point tunes IR theory to z = 2.
  - SUSY isn't necessary: just need massless adjoint fermions.

Fundamental matter

If we add $n_F \ll$ N fundamental quarks, have to pick their BCs.

$$\psi_a(x_3 + L) = e^{i\alpha_a}\psi_a(x_3)\,,\ a = 1, \ldots, n_{\mathrm{F}}$$

What's effect on emergent dimension story?

- Use index theorems to see how ψ_a couple to σ_i

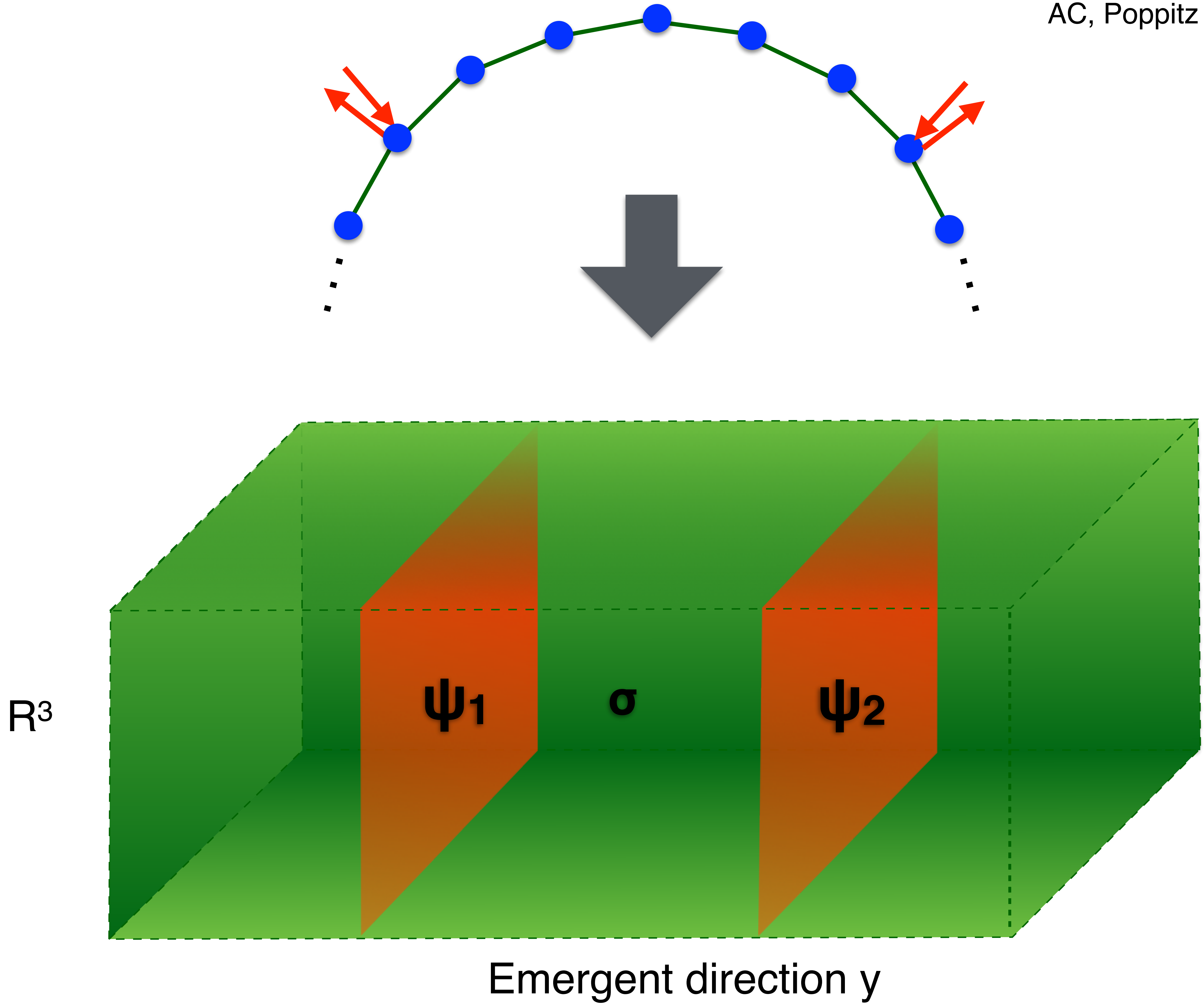
Poppitz, Unsal, 2009; AC, Schafer, Unsal, 2016; AC, Poppitz 2016

Each fundamental quark field brings in two fermion zero modes;
they sit on one of the N monopole-instantons.

- Which monopole gets the fundamental zero modes depends on BCs.
- Result: fundamentals don't propagate into extra dimension!

Fundamental fermions live on 3d branes in a 4d bulk spacetime

AC, Poppitz 2016



Wait, what?

AC, Poppitz 2016

Took close look at rare case of solvable large N limit.

Got **two** startling things.

(1) Long-distance theory is non-trivially scale-invariant.

(2) Put 4D QFT on circle. Pushed circle to be small, but then somehow NP dynamics generated a large circle!

Adjoint matter lives macroscopic 4d bulk, fundamental matter lives on branes.

How reliable are these conclusions?

Why is this happening?

Gaplessness

AC, Poppitz 2016

Is $\sum_a \sigma_a^2$ allowed in long-distance EFT?

No.

Compactness of SU(N) $\Longleftrightarrow$ magnetic charge quantization

Potential only has differences of σ_i, σ_j; generates derivatives!

$$\exp\left(i\left[\sum_k c_k \vec{\alpha}_k\right]\cdot\vec{\sigma}\right)$$

Impossible to get mass term for emergent 4D σ scalar!

$\cos(\vec{\sigma}\cdot\vec{1}) \sim \sum_a \sigma_a^2 \sim \int d^3x\, dy\, \sigma^2$ is forbidden.

Gaplessness

So far neglected “Kahler potential” - in general it isn’t trivial!

$$S_\sigma = \int d^3x \sum_a (\partial_\mu \sigma^a)^2 \to \int d^3x \sum_{a,b} g_{ab}(\lambda) \partial_\mu \sigma^a \partial^\mu \sigma^b$$

Can’t gap out large N theory!

From extra dimension point of view, this is wave function renormalization ⇒ anomalous dimensions!

- Resulting anomalous dimensions vanish as $N L \Lambda \to 0$
- So at large N, get **non-trivial** Lifshitz scale-invariant IR fixed points, which are weakly-coupled when $N L \Lambda \ll 1$.

Interpreting extra dimension

Re-examine how extra dimension appears in e.g. deformed YM case

$$V(\vec{\sigma}) \sim \sum_j (\sigma_j - \sigma_{j+1})^2$$

The mass eigenstates are labeled by the Fourier dual of “color” index j:

$$\sigma_j = \sum_p e^{-2\pi i p j/N} \sigma'_p$$

$$m_p \sim m_W e^{-4\pi^2/\lambda} |\sin(\pi p/N)|$$

But the dual of the index j is the winding number! So the lattice momentum quantum number is the holonomy winding number.

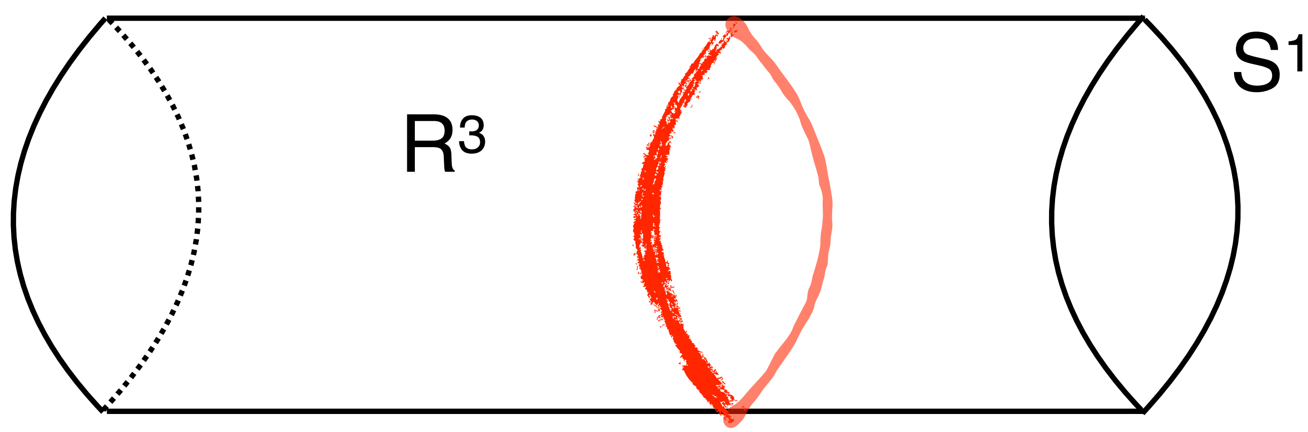
$$\sigma^j \sim F^j_{\mu\nu} \sim \sum_p e^{2\pi i j p/N} \mathrm{tr}\left(\Omega^p F_{\mu\nu}\right)$$

Interpreting extra dimension

Tempting but incomplete interpretation follows.

We took large N confining theory, put it on tiny circle.

Confining string winding modes become light: T-duality



$$E^2 \sim \frac{n^2}{R^2} + \frac{p^2 R^2}{\alpha'^2} \rightarrow \frac{p^2}{\tilde{R}^2}\,, \;\; \tilde{R} = \alpha'/R$$

Interpreting extra dimension

Since "T-dual" dimension comes from confining string, this extra dimension **must** be a discretized one!

- Wind N times = no winding, because baryons aren't confined.
- Upper cutoff on winding = upper cutoff on emergent momentum

$$E^2 \sim \frac{n^2}{R^2} + \frac{p^2 R^2}{\alpha'^2} \to \frac{p^2}{\tilde{R}^2}\,, \;\; \tilde{R} = \alpha'/R$$

This formula only valid for $p \ll N$.

This is of course just what we see!

Interpreting extra dimension

Truth in advertising:

Marketoonist.com

Problems with T-duality interpretation

Truth in advertising:

- No stringy interpretation of the Lifshitz scaling.
- Relating size of extra dimension to string tension confusing.
  - At small L, R^3 string tension ≠ S^1 string tension.

$$L\tilde{L} \sim \alpha'_{\mathbb{R}^3}$$

$$L\tilde{L} \sim N\alpha'_{S^1}$$

Extra factor of N unexpected from T-duality

- But S^1 strings are very short, so not best definition of α'_{S^1} isn't obvious
- T-duality interpretation very tempting, but not proven.

Gaplessness

Big open question: what is the IR behavior when NLΛ $\gg$ 1?

- Naively expect smooth behavior in $\eta = NL\Lambda$, with volume independence setting in smoothly for large η
- Three possibilities occur to me:
  - Behavior really is smooth - then the theory has a gapless sector even for large η!
  - Large N phase transition at some critical η to a gapped phase
  - T-duality picture is actually correct, and somehow allows physics to be smooth in η, without implying large η gaplessness

Curvature?

- The emergent dimension is flat when center symmetry is unbroken
- We could make center symmetry break spontaneously or explicitly.
  - Does the extra dimension pick a warp factor, so that the emergent spacetime becomes curved?
  - Work in progress!

AC, Andy Sheng, Baiyang Zhang, 2020?

Conclusions

Yang-Mills and QCD still surprise us a lot.

Weak-coupling insight into confinement dynamics in 4d

New insights into renormalons

Working examples of large N volume independence

Remarkable Bose-Fermi correlations without supersymmetry

Emergent dimension within field theory

Gapless confining theory

Fundamental fields ↔ branes in an emergent bulk,
all from QFT

Lots more to understand…

Thank you for your attention!