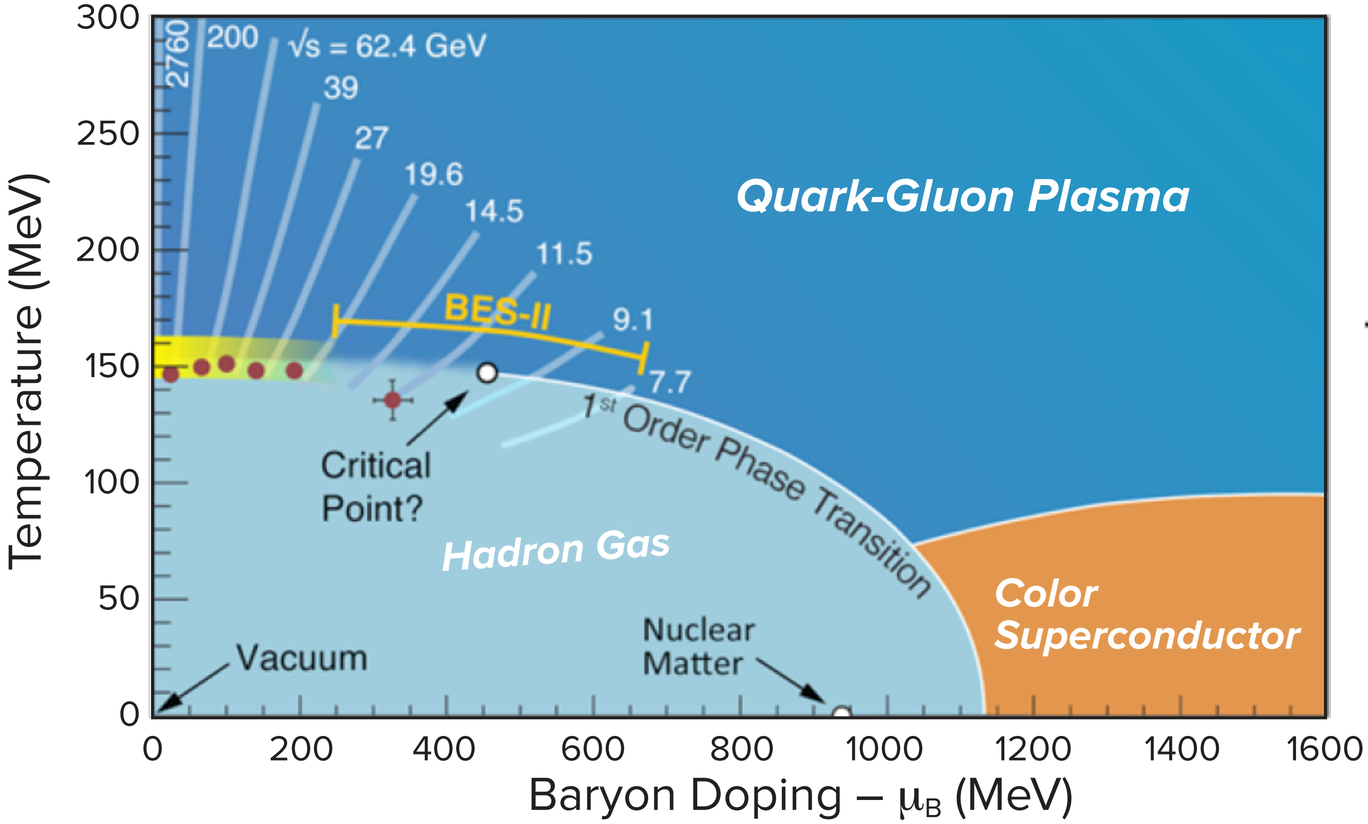
Resurgence, Phase Transitions and Large N

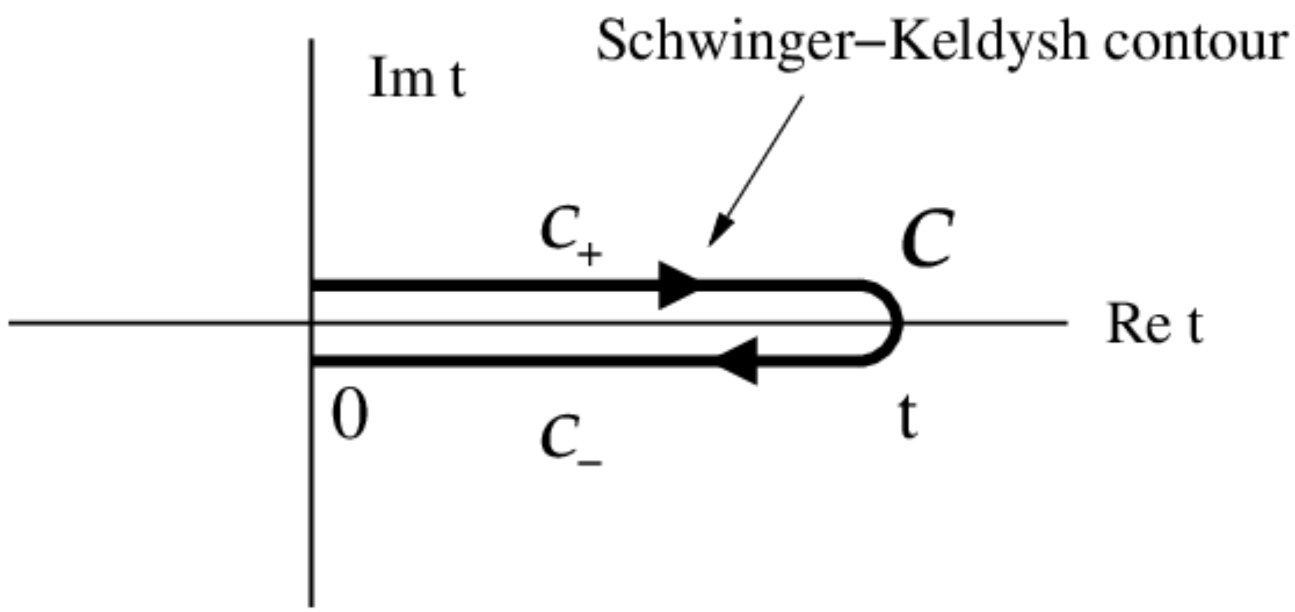
Gerald Dunne
University of Connecticut

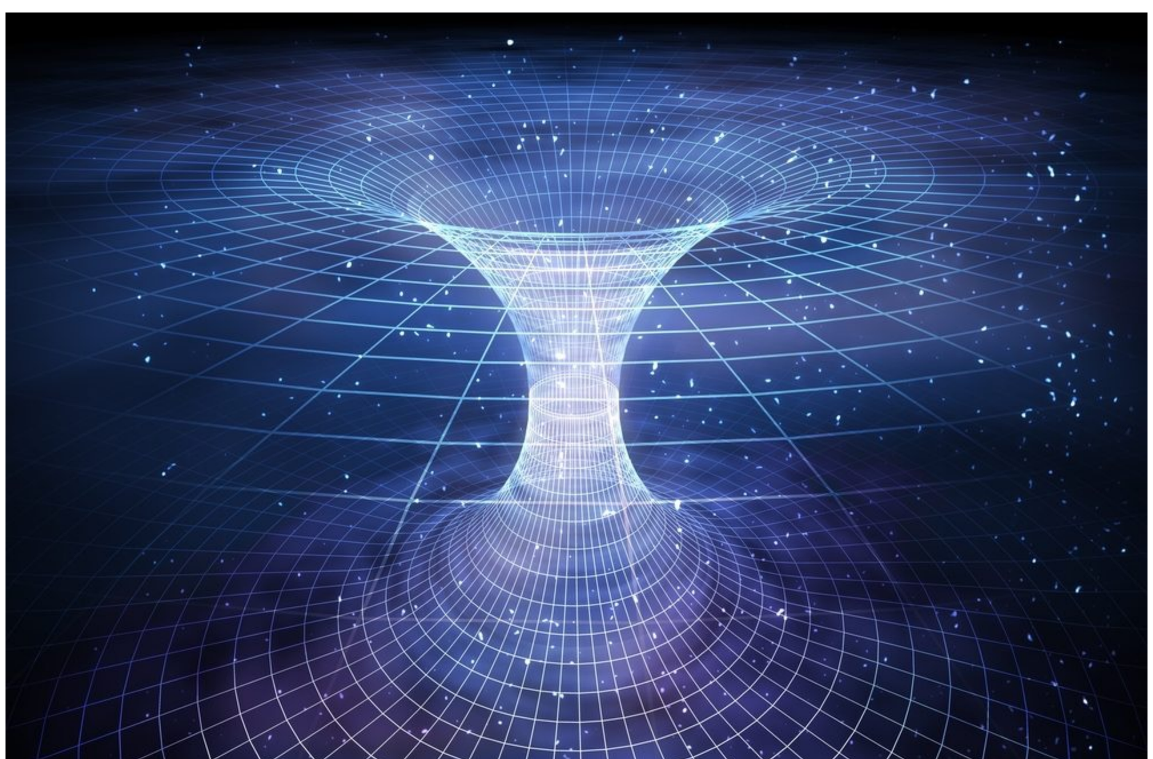
Yukawa Institute/RIKEN iTHEMS Conference 2020 , September 2020

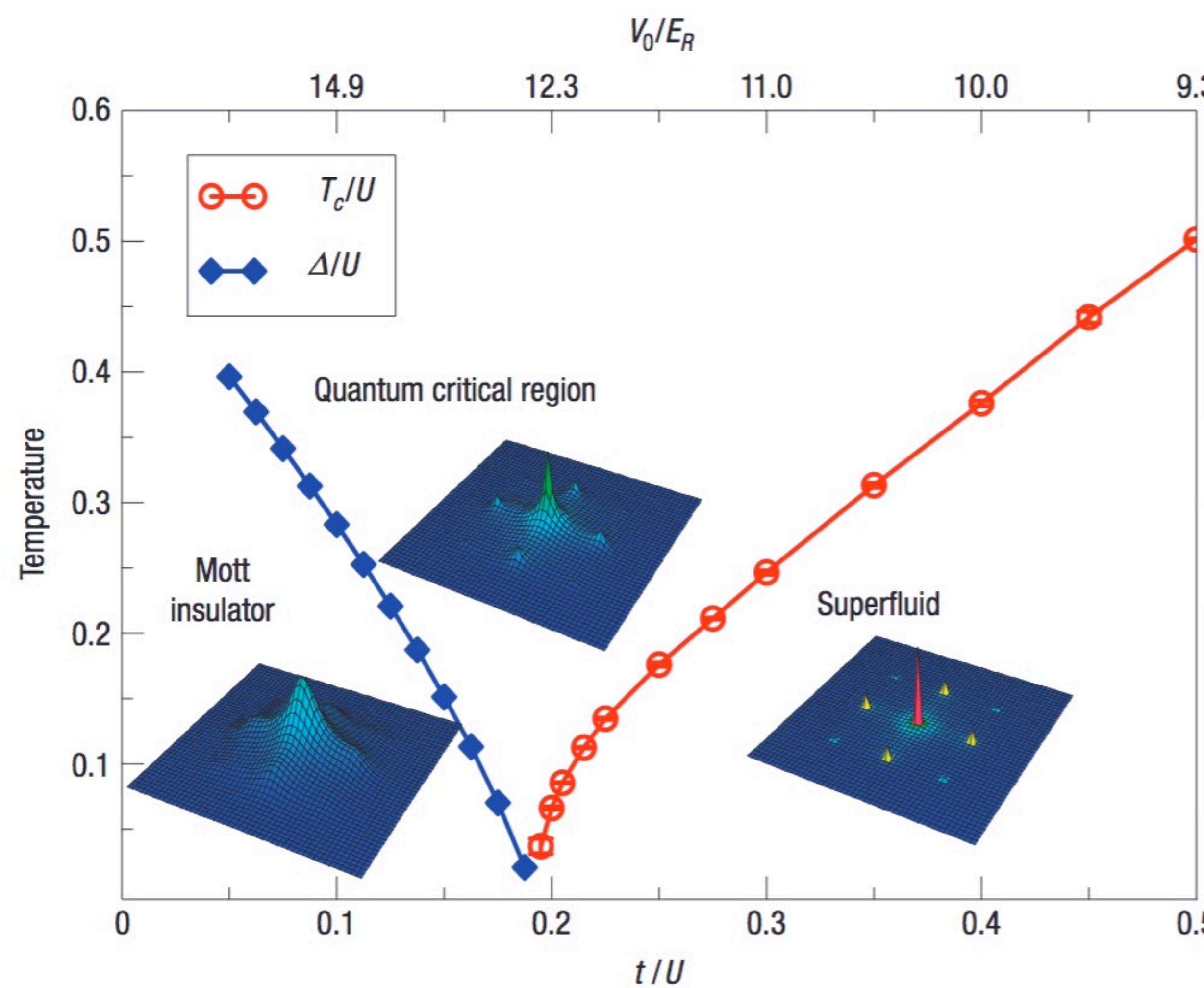
review: arXiv:1601.03414 ; winter school lectures

Physical Motivation









Physical Motivation: Quantum Physics in Extreme Conditions

- QCD phase diagram
- non-equilibrium physics at strong-coupling
- (quantum) phase transitions in cold atom systems
- quantum systems in extreme background fields
- transition to hydrodynamics
- quantum gravity

extreme systems are extremely difficult to analyze quantitatively

extreme = strongly-coupled; high density; ultra-fast driving; ultra-cold; strong fields; strong curvature; heavy ion collisions; …

- perturbation theory is of limited use
- non-perturbative semi-classical methods: “instantons"
- non-perturbative numerical methods: Monte Carlo
- asymptotics

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"resurgence": new form of asymptotics that unifies these approaches

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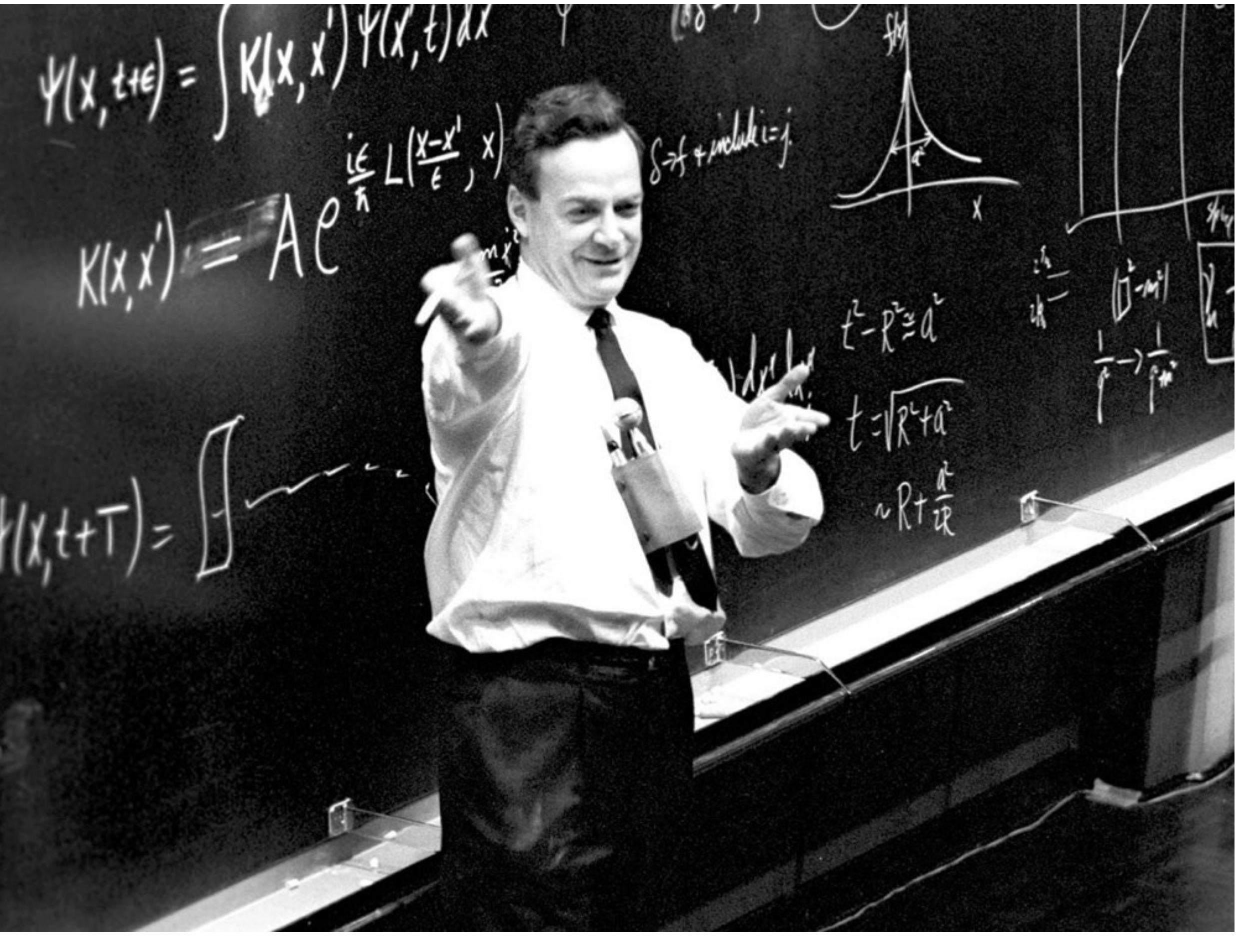
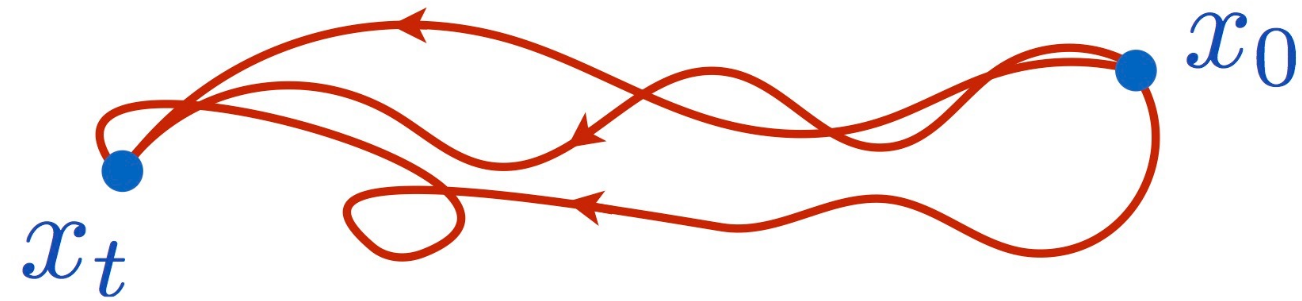
technical problem: what does a quantum path integral really mean?

The Feynman Path Integral



$$\langle x_t | e^{-i\hat{H}t/\hbar} | x_0 \rangle =$$

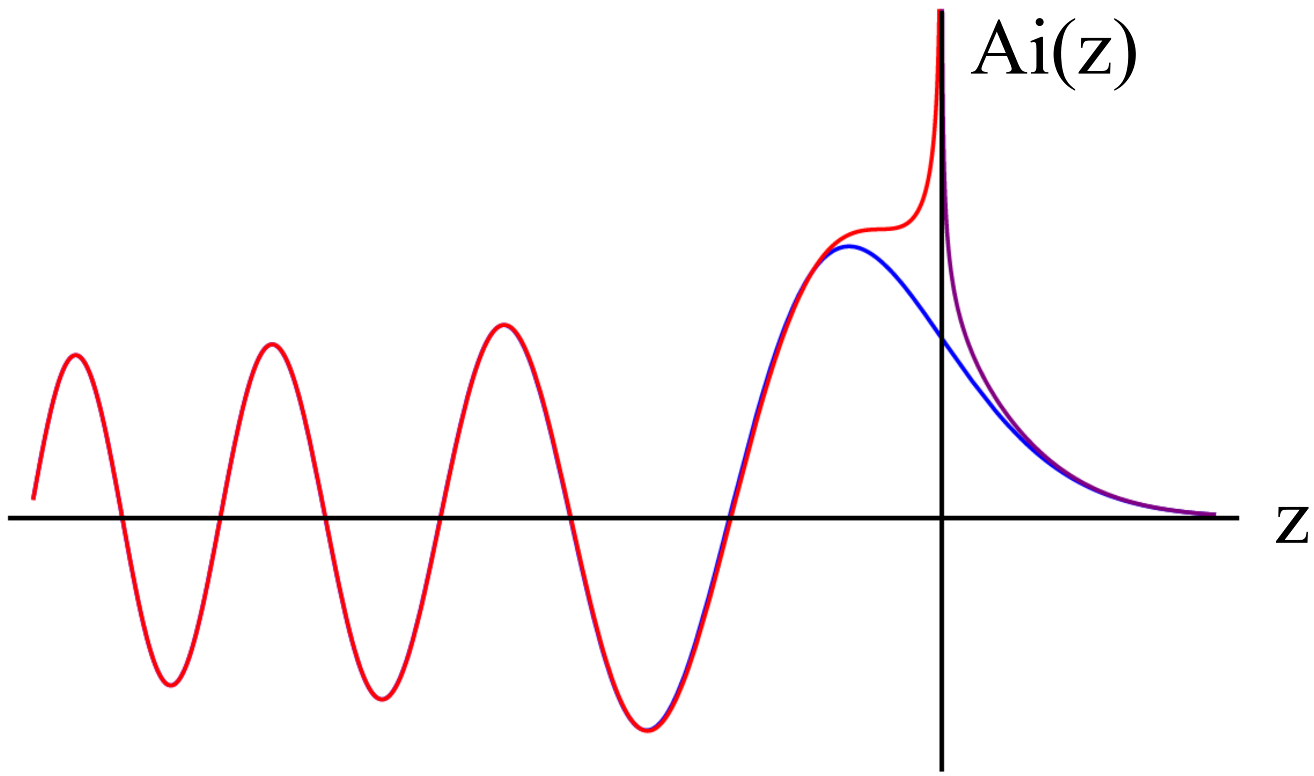


QM: $$\int \mathcal{D}x(t)\, \exp\left[\frac{i}{\hbar} S[x(t)]\right]$$

QFT: $$\int \mathcal{D}A(x^\mu)\, \exp\left[\frac{i}{g^2} S\left[A(x^\mu)\right]\right]$$

- stationary phase approximation: classical physics
- quantum perturbation theory: fluctuations about trivial saddle point
- other saddle points: non-perturbative physics
- resurgence: saddle points are related by analytic continuation, so perturbative and non-perturbative physics are *unified*

Stokes and the Airy Function: "Stokes Phenomenon"



$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\left(\frac{1}{3}x^3 + z\,x\right)}dx \sim \begin{cases} \dfrac{e^{-\frac{2}{3}z^{3/2}}}{2\sqrt{\pi}\,z^{1/4}} & , \quad z \to +\infty \\ \\ \dfrac{\sin\left(\frac{2}{3}(-z)^{3/2} + \frac{\pi}{4}\right)}{\sqrt{\pi}\,(-z)^{1/4}} & , \quad z \to -\infty \end{cases}$$

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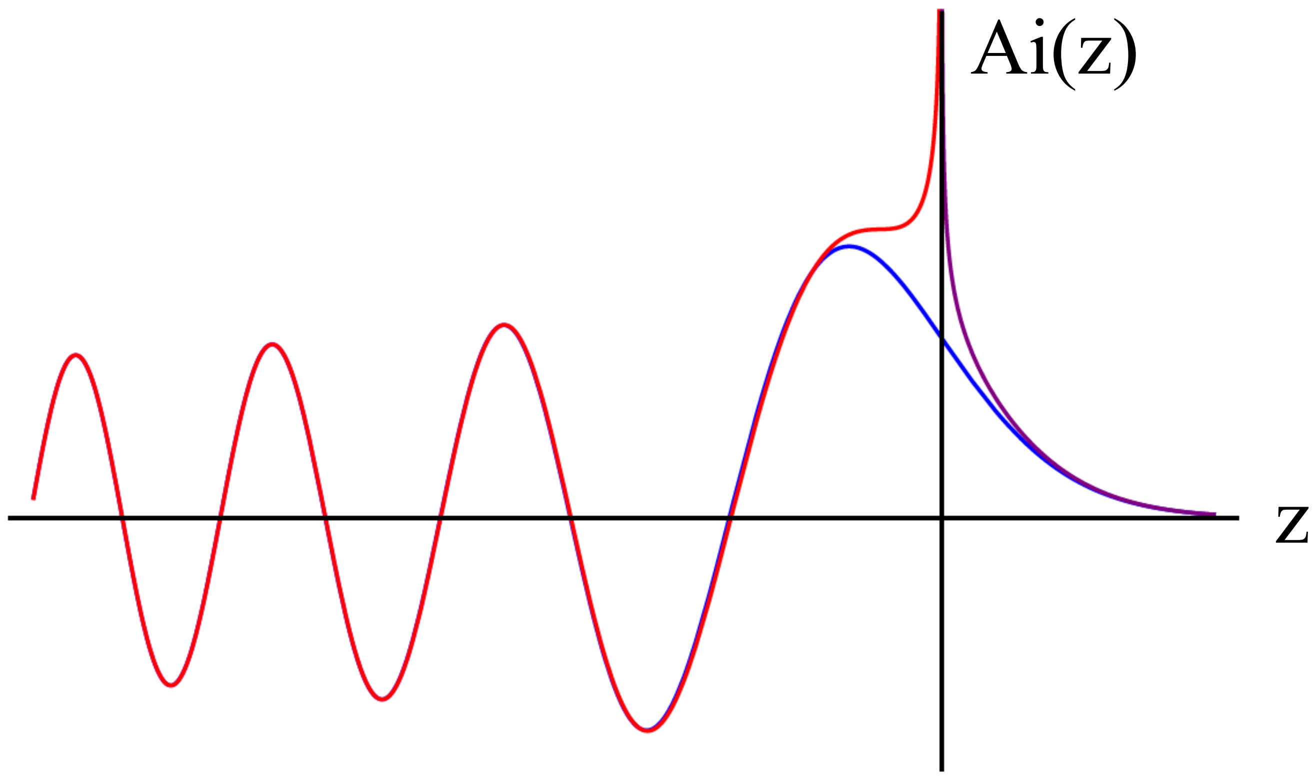

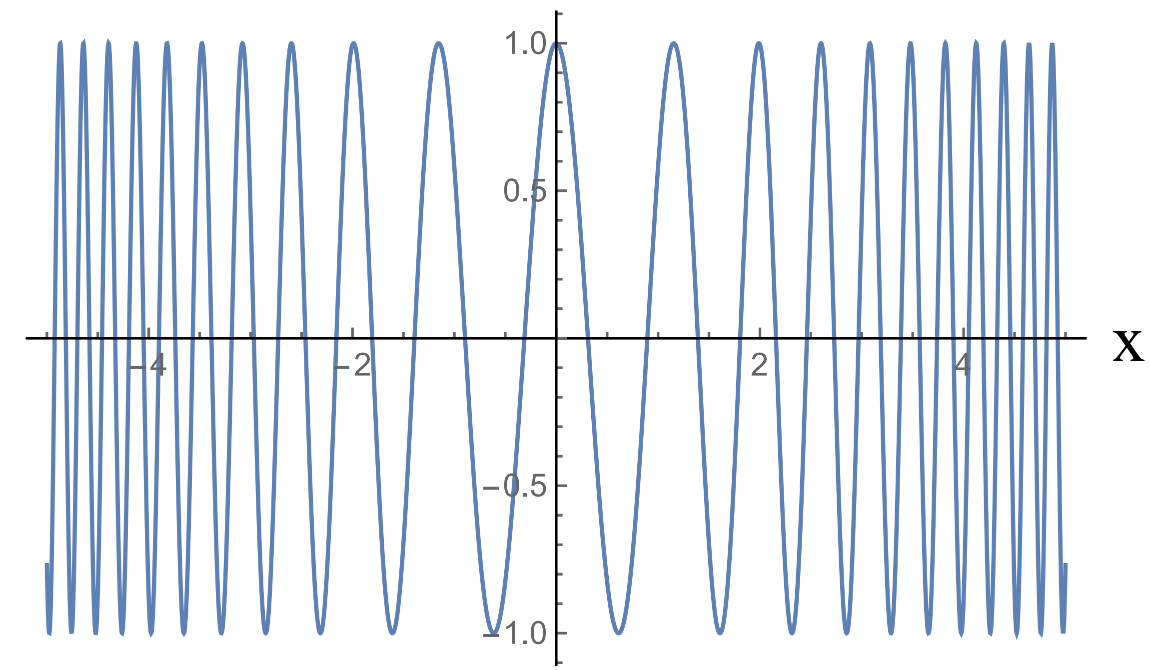

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- integral <u>cannot</u> be evaluated without contour deformation
- "Stokes transition" at z=0
- fluctuation expansions about saddles <u>must</u> be divergent, and <u>must</u> be related
- underlies optics and WKB analysis

Analytic Continuation of Path Integrals

since we need complex analysis and contour deformation to make sense of oscillatory integrals, it is natural to explore similar methods for (infinite dimensional) path integrals

$$\int \mathcal{D}x(t)\, \exp\left[\frac{i}{\hbar}\, S[x(t)]\right] \longleftrightarrow \int \mathcal{D}x(t)\, \exp\left[-\frac{1}{\hbar}\, S[x(t)]\right]$$

idea: seek a computationally viable constructive definition of the path integral as a resurgent trans-series

Resurgent Trans-Series

resurgence: "new" idea in mathematics

Dingle 1960s, Ecalle, 1980s; Stokes 1850

perturbative series $\longrightarrow$ "trans-series"

$$f(\hbar) = \sum_p c_{[p]}\, \hbar^p \quad \longrightarrow \quad f(\hbar) = \sum_k \sum_p \sum_l c_{[kpl]}\, e^{-\frac{k}{\hbar}}\, \hbar^p \, (\ln \hbar)^l$$

physics: • unifies perturbative and non-perturbative physics

mathematics:
- trans-series is well-defined under analytic continuation
- expansions about different saddles are related
- exponentially improved asymptotics

Resurgent Functions

"resurgent functions display at each of their singular points a behaviour closely related to their behaviour at the origin. Loosely speaking, these functions resurrect, or surge up - in a slightly different guise, as it were - at their singularities"

J. Ecalle, 1980

conjecture: this structure occurs for all "natural" problems

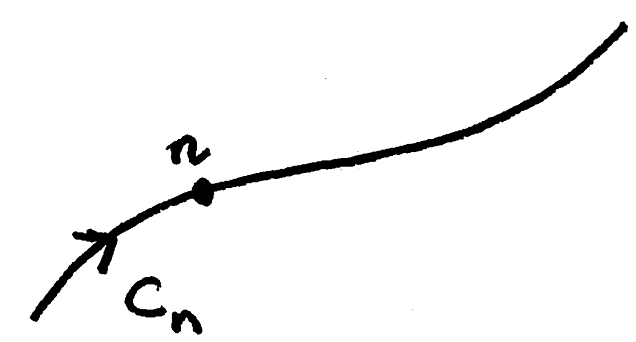
Resurgence in Exponential Integrals

steepest descent integral through saddle point "n":

$$I^{(n)}(\hbar) = \int_{C_n} dx\, e^{\frac{i}{\hbar}\, f(x)} = \frac{1}{\sqrt{1/\hbar}}\, e^{\frac{i}{\hbar}\, f_n}\, T^{(n)}(\hbar)$$

all fluctuations beyond the Gaussian approximation:

$$T^{(n)}(\hbar) \sim \sum_{r=0}^{\infty} T_r^{(n)}\, \hbar^r$$



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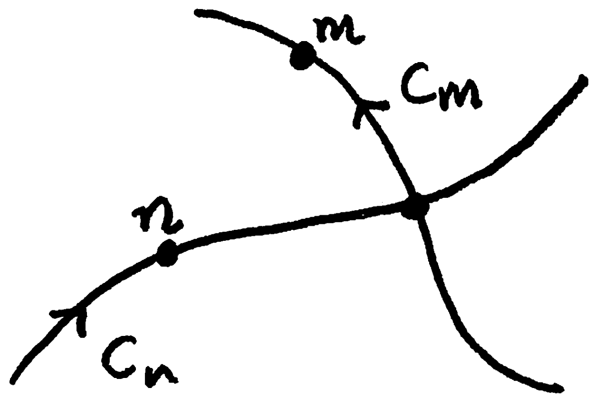

straightforward complex analysis implies:

<u>universal</u> large orders of fluctuation coefficients: $(F_{nm} \equiv f_m - f_n)$

$$T_r^{(n)} \sim \frac{(r-1)!}{2\pi\, i} \sum_m \frac{(\pm 1)}{(F_{nm})^r} \left[T_0^{(m)} + \frac{F_{nm}}{(r-1)}\, T_1^{(m)} + \frac{(F_{nm})^2}{(r-1)(r-2)}\, T_2^{(m)} + \ldots \right]$$

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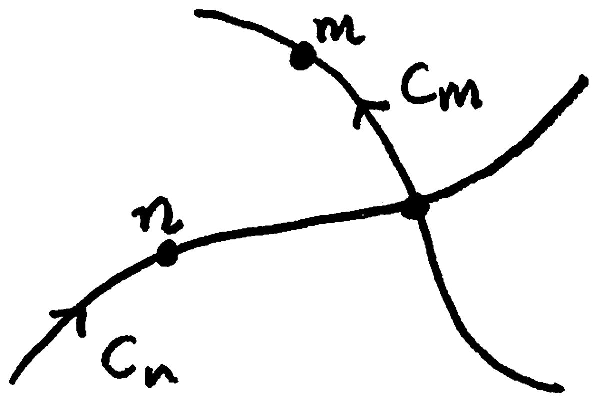

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fluctuations about different saddles are quantitatively related

Resurgence in Exponential Integrals

canonical example: Airy function: 2 saddle points

$$T_r^{\pm} = (\pm 1)^r \frac{\Gamma\left(r+\frac{1}{6}\right)\Gamma\left(r+\frac{5}{6}\right)}{(2\pi)\left(\frac{4}{3}\right)^r r!} = \left\{1, \pm\frac{5}{48}, \frac{385}{4608}, \pm\frac{85085}{663552}, \dots\right\}$$

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large orders of fluctuation coefficients:

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generic "large-order/low-order" resurgence relation

Resurgence in Exponential Integrals

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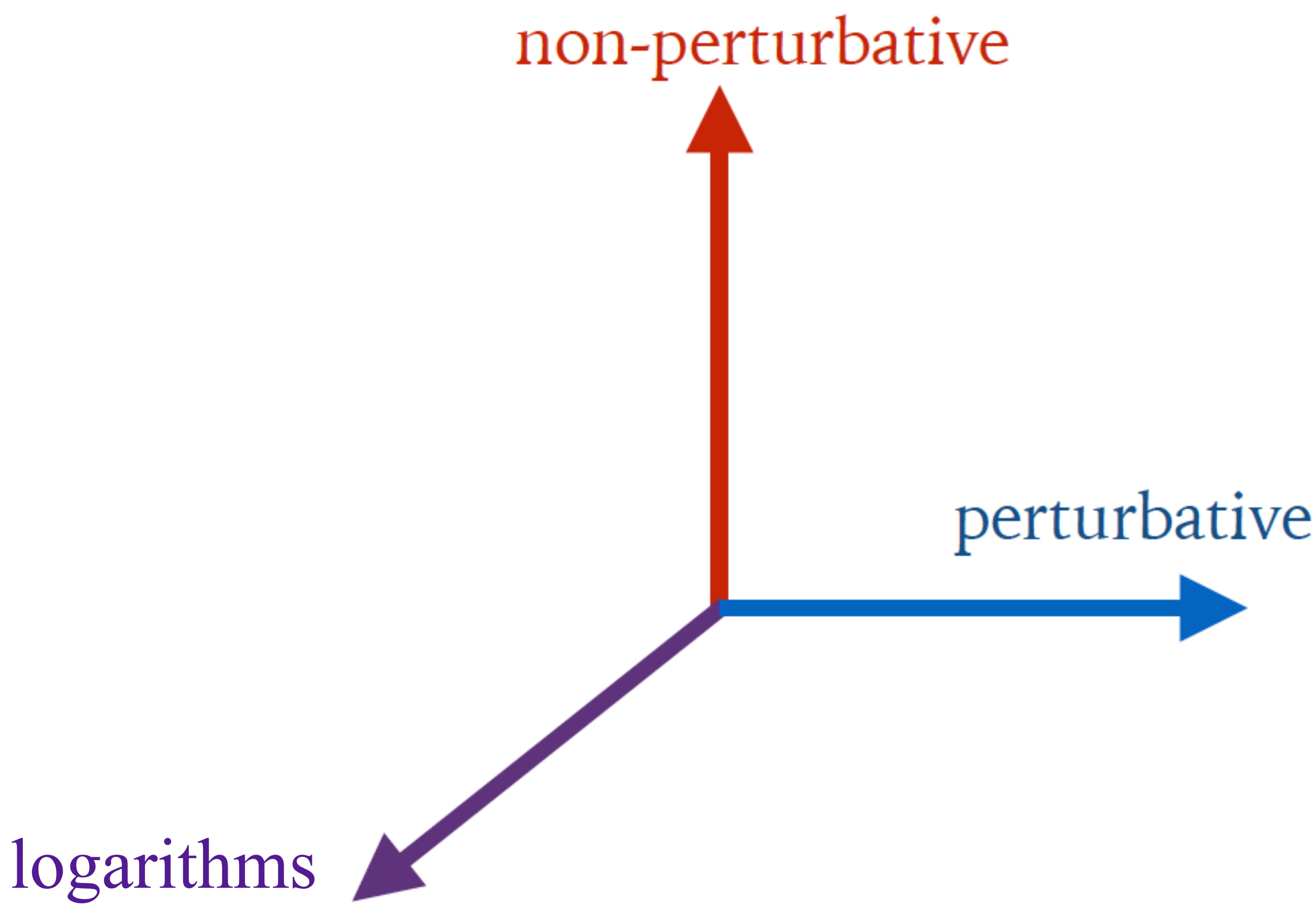
generic "large-order/low-order" resurgence relation

amazing fact: this large-order/low-order behavior has been found in matrix models, QM, QFT, string theory, …

the only natural way to explain this is via analytic continuation of path integrals

Decoding a Path Integral as a Trans-Series

$$\int \mathcal{D}A\, e^{\frac{i}{\hbar}S[A]} = \sum_{\text{thimbles}} e^{\frac{i}{\hbar}S[A_{\text{thimble}}]} \times (\text{fluctuations}) \times (\text{logs})$$



- expansions along different axes must be <u>quantitatively</u> related
- expansions about different saddles must be <u>quantitatively</u> related

Perturbation Theory

perturbation theory works, but it is generically divergent

this is actually a very good thing !

and there is a lot of interesting physics behind this

PHYSICAL REVIEW VOLUME 85, NUMBER 4 FEBRUARY 15, 1952

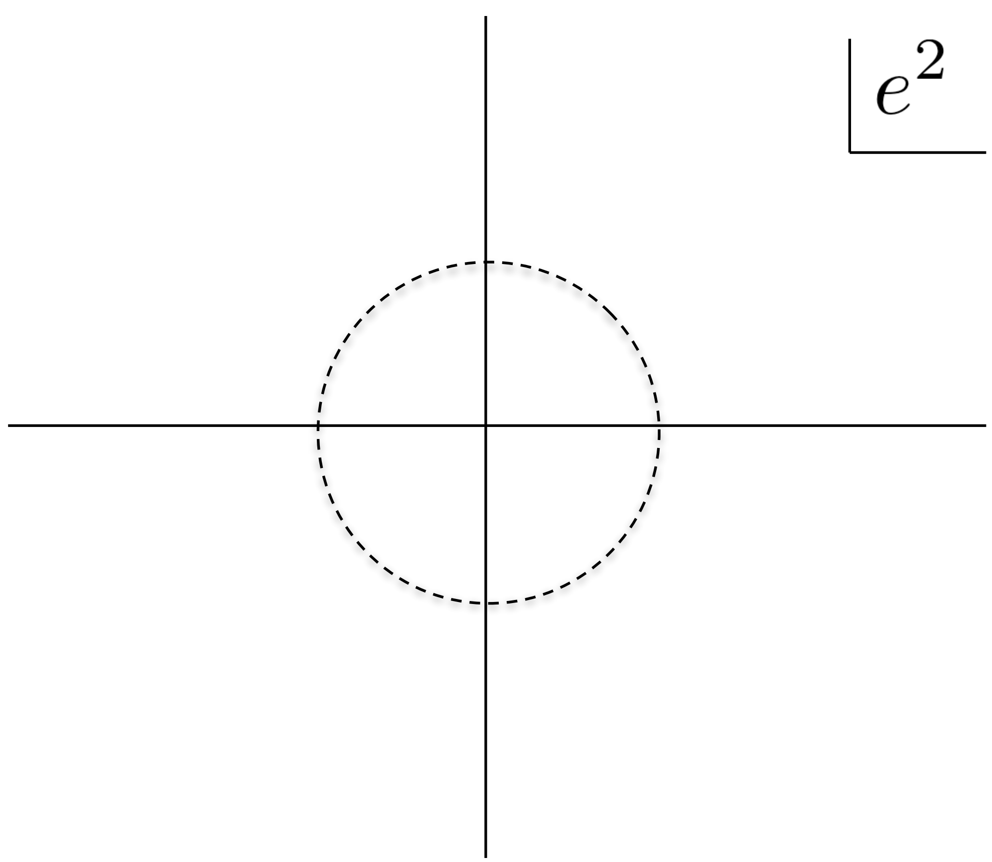
Divergence of Perturbation Theory in Quantum Electrodynamics

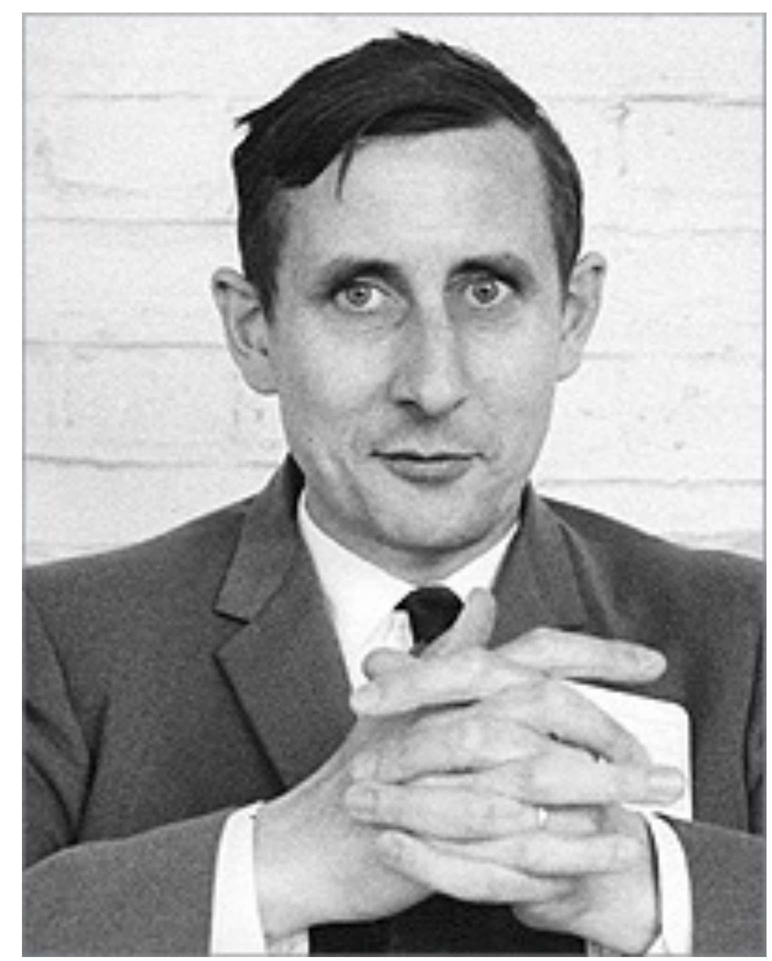
F. J. DYSON
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York

(Received November 5, 1951)

An argument is presented which leads tentatively to the conclusion that all the power-series expansions currently in use in quantum electrodynamics are divergent after the renormalization of mass and charge. The divergence in no way restricts the accuracy of practical calculations that can be made with the theory, but raises important questions of principle concerning the nature of the physical concepts upon which the theory is built.

$$F = a_0 + a_1 e^2 + a_2 e^4 + a_3 e^6 + \dots$$





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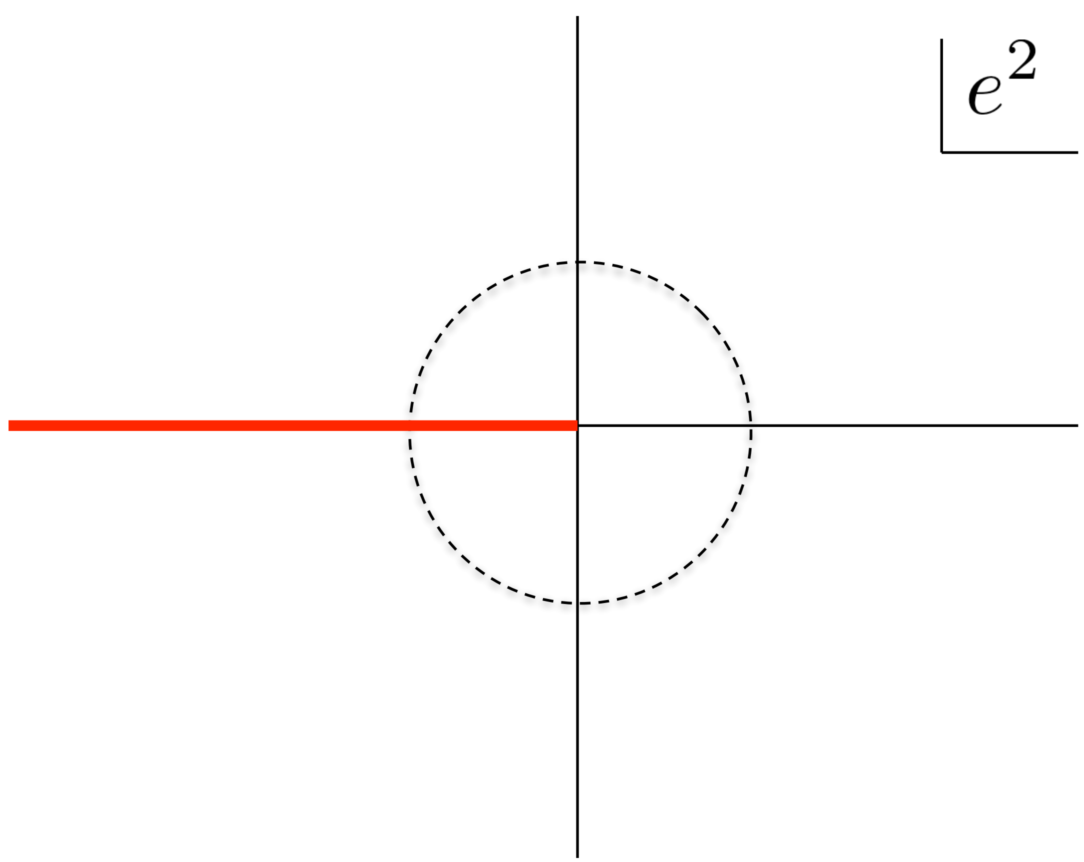
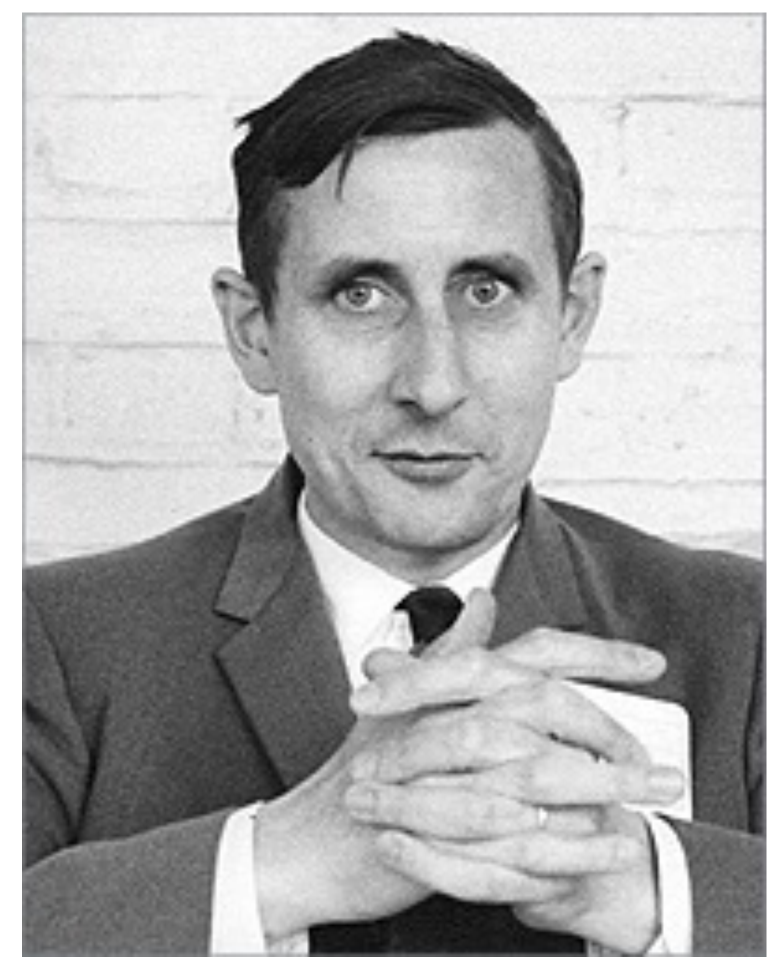


$e^2 < 0$

unstable



The Struggle to Make Sense of Divergent Series

205

DE
SERIEBVS DIVERGENTIBVS.

Auctore LEON. EVLERO.

§. 1.

$$\sum_{n=0}^{\infty} (-1)^n \, n! \, x^n = ???$$

L. Euler, *De seriebus divergentibus*, Opera Omnia, I, 14, 585–617, 1760.

The Struggle to Make Sense of Divergent Series

"Borel summation"

factorial: $n! = \int_0^\infty dt\, e^{-t}\, t^n$

$$f(x) = \sum_{n=0}^{\infty} (-1)^n n!\, x^n = \int_0^\infty dt\, e^{-t} \frac{1}{1 + x\, t}$$

convergent for all x > 0

Emile Borel

The Struggle to Make Sense of Divergent Series

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convergent for all x > 0

$$f(-x) = \sum_{n=0}^{\infty} n!\, x^n = \int_0^\infty dt\, e^{-t} \frac{1}{1-x\,t}$$

$\Rightarrow$ $\mathrm{Im}[f(-x)] \sim e^{-1/x}$

nonperturbative imaginary part !

Emile Borel

QM Perturbation Theory: Zeeman & Stark Effects

Zeeman : divergent, alternating, asymptotic series

$$a_n \sim (-1)^n (2n)!$$

physics: magnetic field causes energy level shifts (real)

Stark : divergent, non-alternating, asymptotic series

$$a_n \sim (2n)!$$

physics:
- electric field causes energy level shifts (real)
- <u>and</u> ionization (imaginary, exponentially small)

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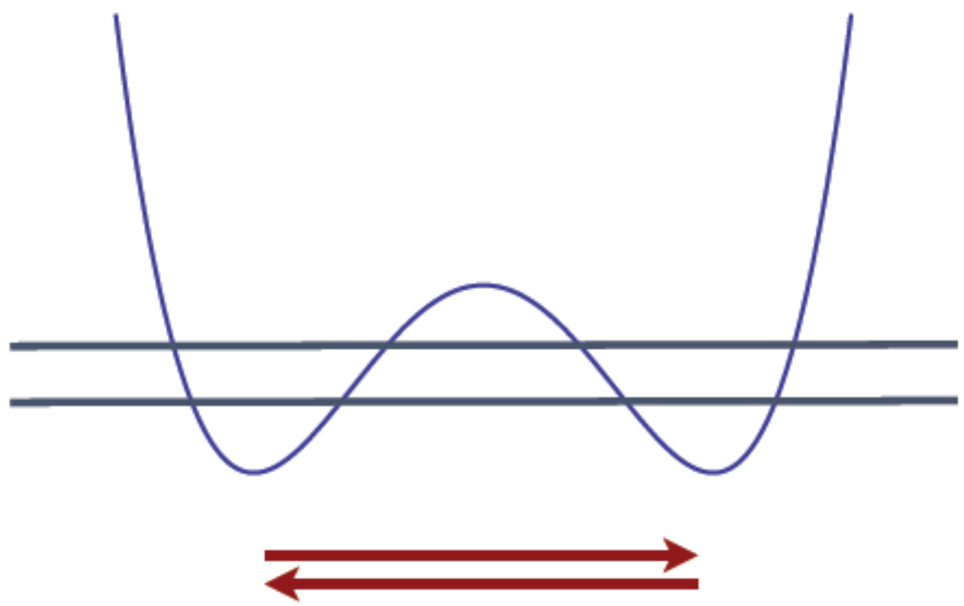
appears nicely consistent with Borel summation approach …

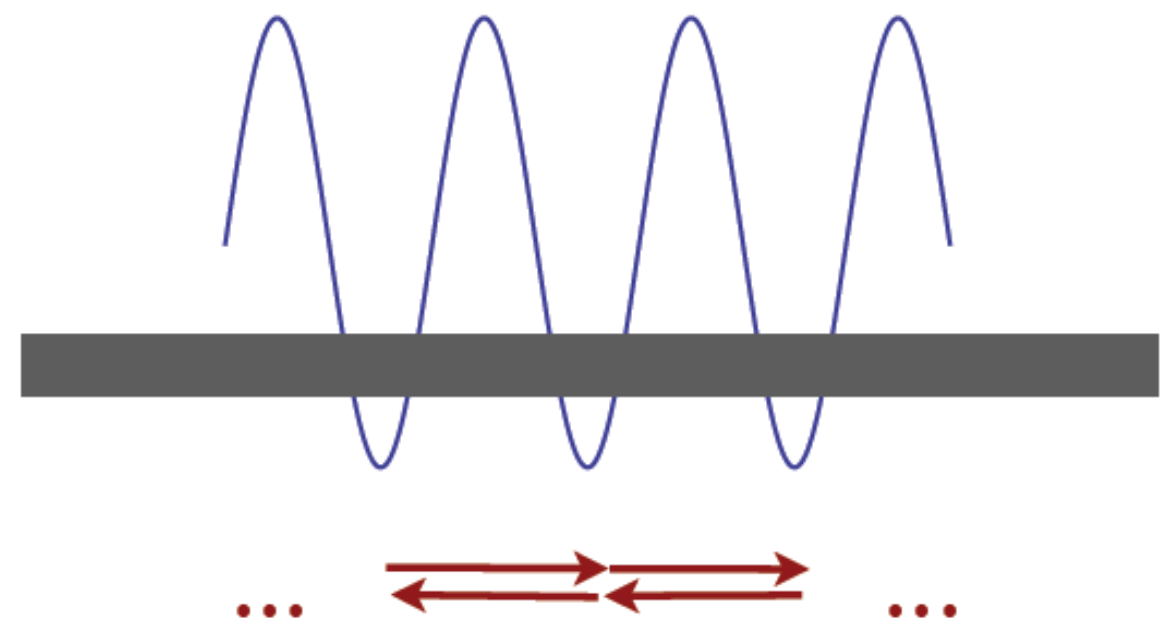
but not so fast …

the story becomes even more interesting …

Instantons and Non-Perturbative Physics



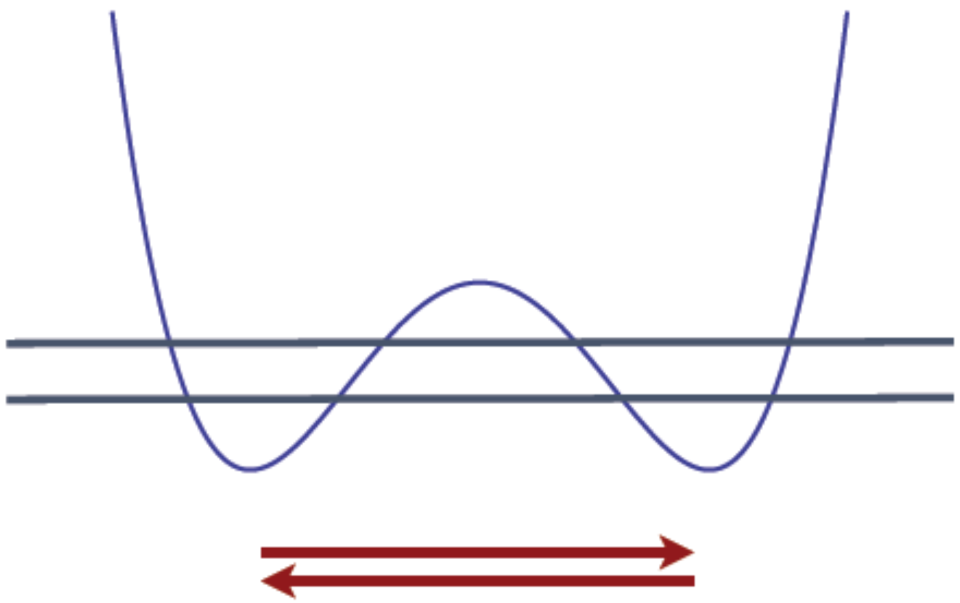
(phase transitions)



(band structure)

- exponentially small non-perturbative splitting due to tunneling
- Yang-Mills theory and QCD have aspects of <u>both</u> systems
- physics of optical lattices and condensates

Instantons and Non-Perturbative Physics



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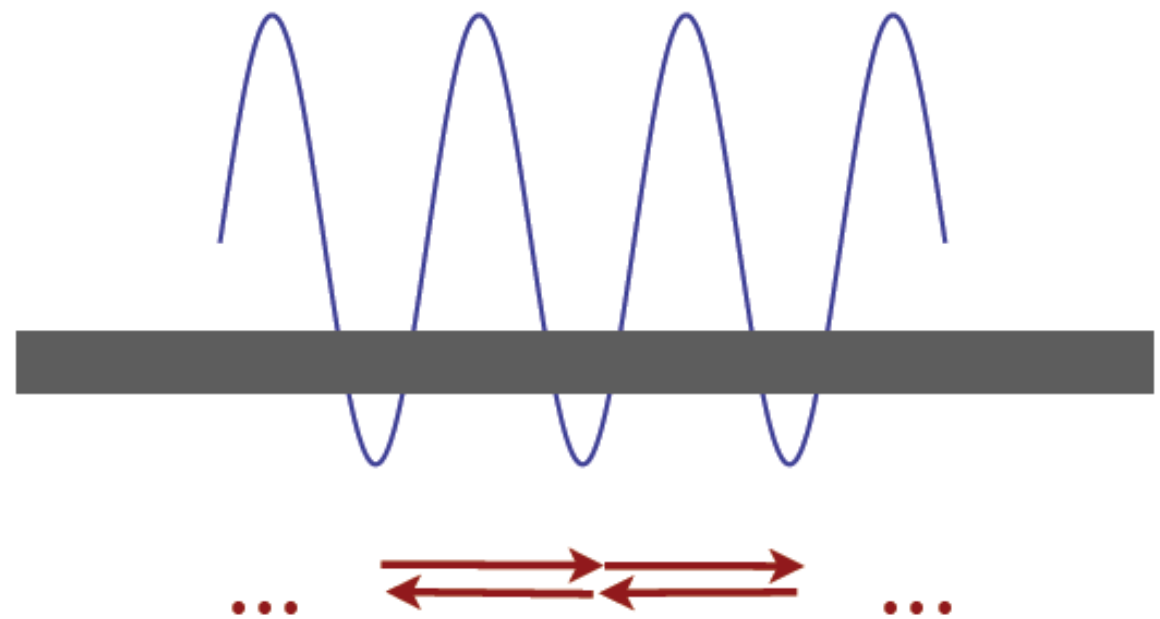

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surprise: perturbation theory is non-alternating divergent !

but these systems are stable ???

A Brilliant Resolution: "BZJ Cancelation Mechanism"

E. B. Bogomolny, 1980; J. Zinn-Justin et al, 1980

perturbation theory + Borel: $\longrightarrow$ $+i\,\exp\left[-\frac{2\,S_I}{\hbar}\right]$

non-perturbative instanton
& anti-instanton interaction: $\longrightarrow$ $-i\,\exp\left[-\frac{2\,S_I}{\hbar}\right]$

unphysical imaginary parts exactly cancel !

separately, each of the perturbative and non-perturbative computations is inconsistent; but combined as a trans-series they are consistent

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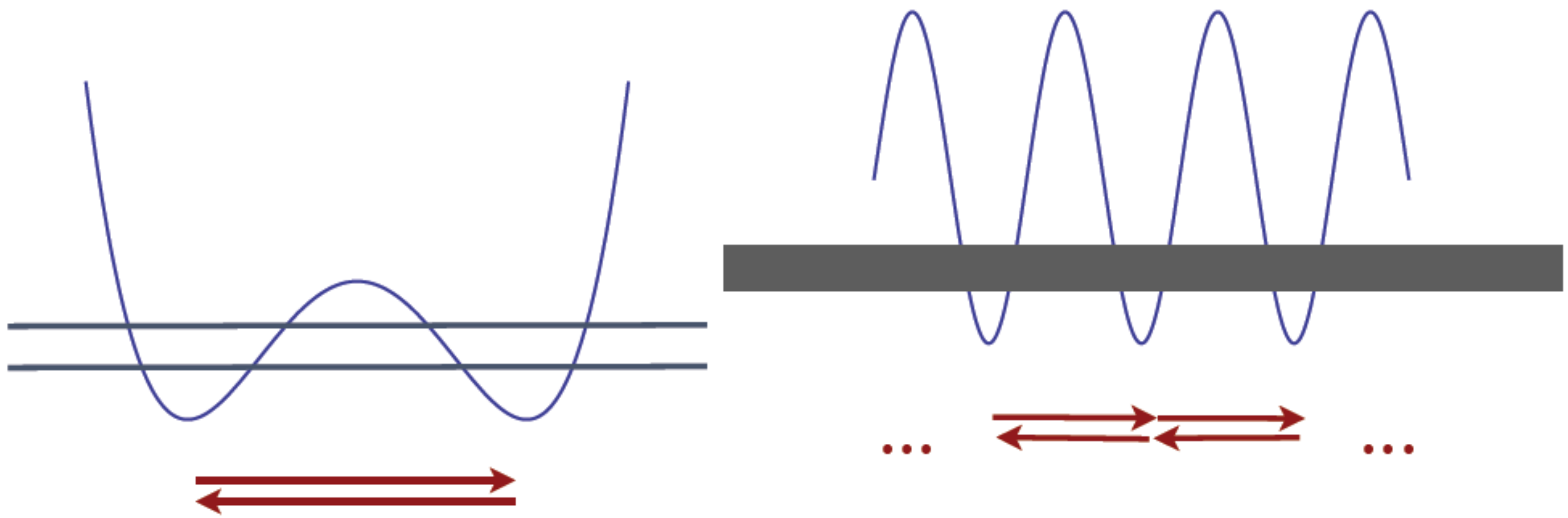
tip-of-the-iceberg: perturbative/non-perturbative relations

"Resurgence": cancelations occur to all orders; the trans-series expression for the energy is real & well-defined

Resurgence in Quantum Mechanical Instanton Models



trans-series for energy, including non-perturbative splitting:

$$E_{\pm}(\hbar, N) = E_{\rm pert}(\hbar, N) \pm \frac{\hbar}{\sqrt{2\pi}} \frac{1}{N!} \left(\frac{32}{\hbar}\right)^{N+\frac{1}{2}} \exp\left[-\frac{8}{\hbar}\right] \mathcal{P}_{\rm inst}(\hbar, N) + \dots$$

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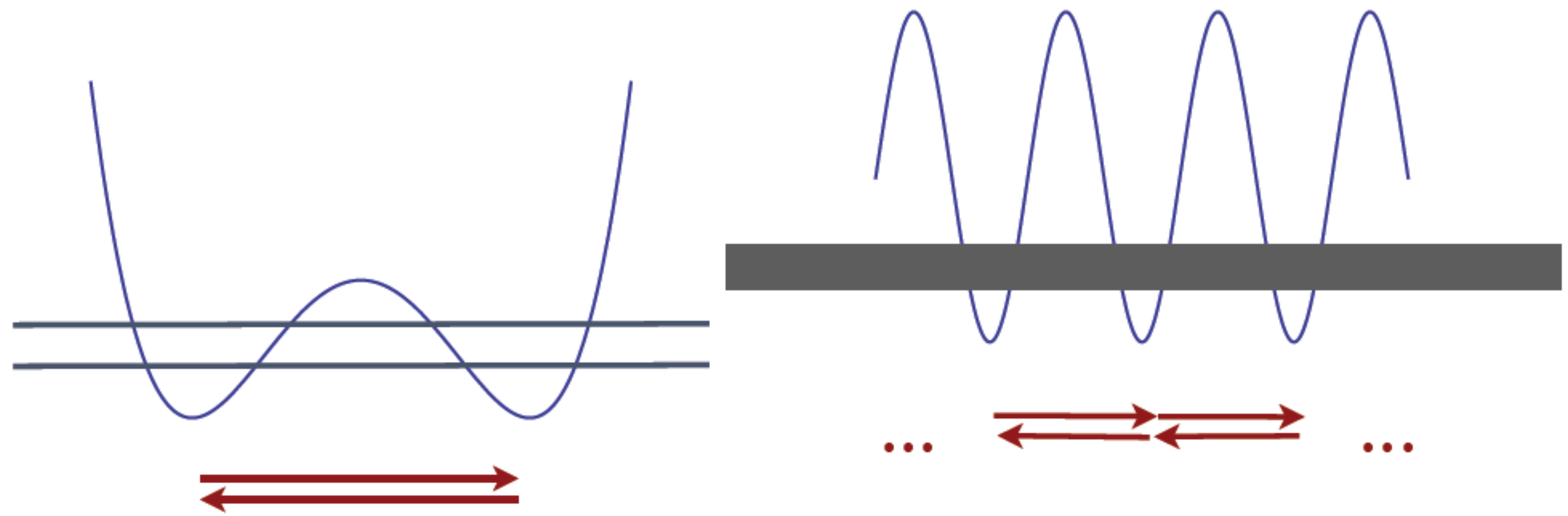

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fluctuations about first non-trivial saddle:

$$\mathcal{P}_{\rm inst}(\hbar, N) = \frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} \exp\left[S \int_0^{\hbar} \frac{d\hbar}{\hbar^3} \left(\frac{\partial E_{\rm pert}(\hbar, N)}{\partial N} - \hbar + \frac{\left(N + \frac{1}{2}\right) \hbar^2}{S}\right)\right]$$

perturbation theory encodes everything … to all orders

Resurgent Functions

occurs in QM path integrals with an infinite number of saddles

Parametric Resurgence and Phase Transitions

$$Z(\hbar) = \int \mathcal{D}A \, \exp\left(\frac{i}{\hbar} S[A]\right)$$

- in general, we are interested in many parameters

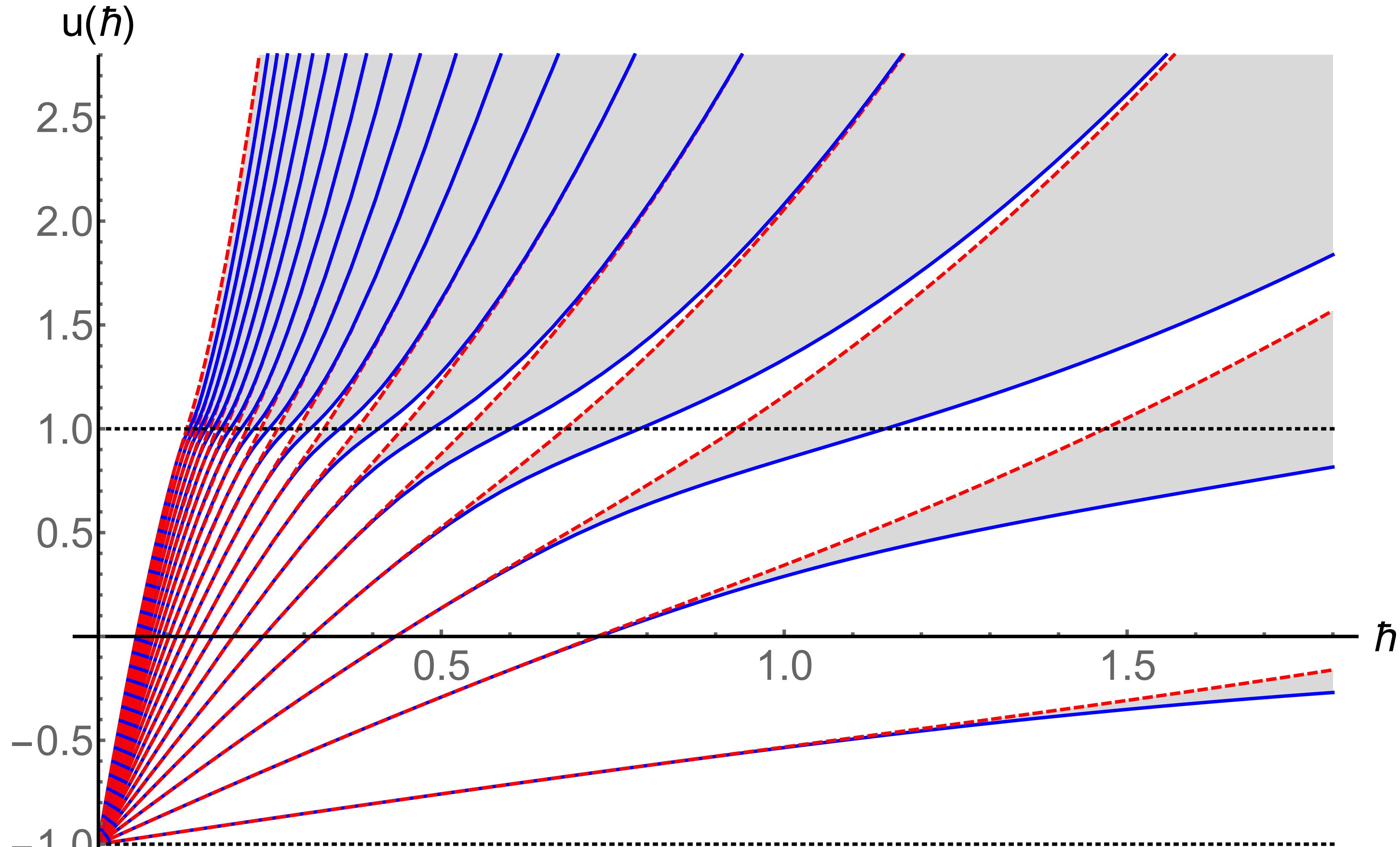
$$Z(\hbar) \;\rightarrow\; Z(\hbar, \text{masses}, \text{couplings}, \mu, T, B, \ldots)$$

- for a phase transition: large N ``thermodynamic limit''

$$Z(\hbar) \;\rightarrow\; Z(\hbar, N), \text{ and } N \rightarrow \infty$$

- multiple parameters: different limits are possible
- "uniform" 't Hooft limit: $N \rightarrow \infty \;,\; \hbar \rightarrow 0 \;:\; \hbar\, N = \text{fixed}$
- trans-series transmutes into different form in the large N limit
- hallmark of a phase transition

Phase Transition in the Mathieu Equation Spectrum



- N= band/gap label; $\hbar$=coupling
- phase transition: narrow bands vs. narrow gaps: $\hbar\, N = \frac{8}{\pi}$
- real instantons vs. complex instantons
- phase transition = "instanton condensation"
- universal phase transition

Basar, GD, 1501.05671,
GD, Unsal, 1603.04924

Resurgence in QFT: Euler-Heisenberg Effective Action

Folgerungen aus der Diracschen Theorie des Positrons.

Von **W. Heisenberg** und **H. Euler** in Leipzig.

Mit 2 Abbildungen. (Eingegangen am 22. Dezember 1935.)

Aus der Diracschen Theorie des Positrons folgt, da jedes elektromagnetische Feld zur Paarerzeugung neigt, eine Abänderung der Maxwellschen Gleichungen des Vakuums. Diese Abänderungen werden für den speziellen Fall berechnet, in dem keine wirklichen Elektronen und Positronen vorhanden sind, und in dem sich das Feld auf Strecken der Compton-Wellenlänge nur wenig ändert. Es ergibt sich für das Feld eine Lagrange-Funktion:

$$\mathfrak{L} = \frac{1}{2}(\mathfrak{E}^2-\mathfrak{B}^2) + \frac{e^2}{h c}\int_0^\infty e^{-\eta}\frac{d\eta}{\eta^3}\left\{ i\eta^2(\mathfrak{E}\mathfrak{B})\cdot\frac{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)+\text{konj}}{\cos\left(\frac{\eta}{|\mathfrak{E}_k|}\sqrt{\mathfrak{E}^2-\mathfrak{B}^2+2i(\mathfrak{E}\mathfrak{B})}\right)-\text{konj}} + |\mathfrak{E}_k|^2 + \frac{\eta^2}{3}(\mathfrak{B}^2-\mathfrak{E}^2)\right\}.$$

$$\left(\begin{array}{l}\mathfrak{E},\mathfrak{B}\quad \text{Kraft auf das Elektron.}\\ |\mathfrak{E}_k| = \frac{m^2c^3}{e\hbar} = \frac{1}{„137“}\frac{e}{(e^2/mc^2)^2} = \text{„Kritische Feldstärke“.}\end{array}\right)$$

- integral representation = Borel sum
- analogue of Stark effect ionization and Dyson's argument
- particle production in E field <u>implies</u> series are divergent

Stokes Phase Transition in QFT

- Schwinger effect with monochromatic E field: $E(t) = \mathcal{E}\,\cos(\omega\, t)$
- Keldysh adiabaticity parameter: $\gamma \equiv \dfrac{m\,c\,\omega}{e\,\mathcal{E}}$
- WKB: $\Gamma_{\rm QED} \sim \exp\left[-\pi \dfrac{m^2\,c^3}{e\,\hbar\,\mathcal{E}}\,g(\gamma)\right]$

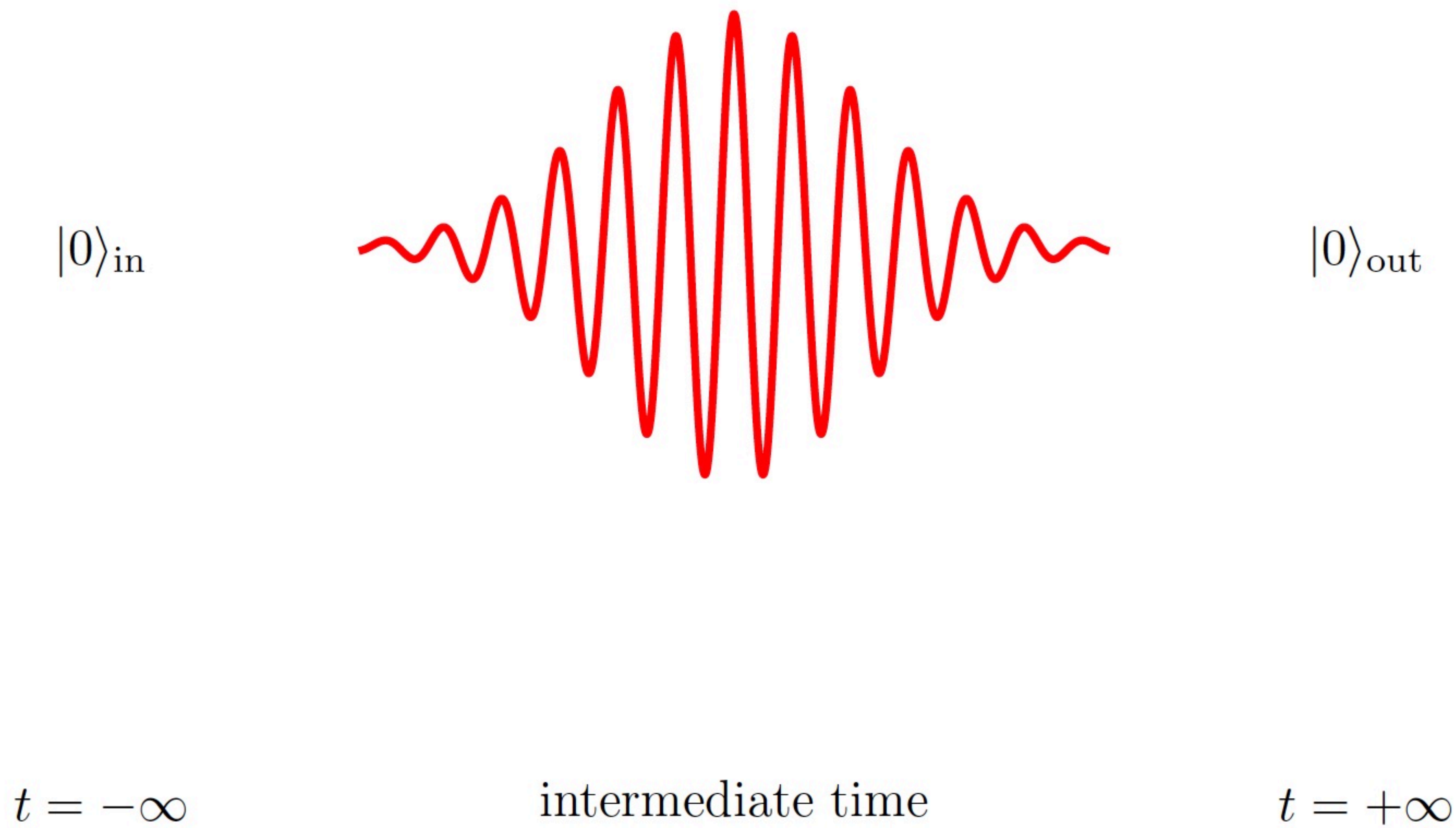
(Keldysh, 1964; Brezin/Itzykson, 1980; Popov, 1981)

$$\Gamma_{\rm QED} \sim \begin{cases} \exp\left[-\pi \frac{m^2\,c^3}{e\,\hbar\,\mathcal{E}}\right] & , \quad \gamma \ll 1 \quad (\text{tunneling}) \\ \left(\frac{e\,\mathcal{E}}{m\,c\,\omega}\right)^{4mc^2/\hbar\omega} & , \quad \gamma \gg 1 \quad (\text{multiphoton}) \end{cases}$$

- phase transition: tunneling vs. multi-photon "ionization"
- phase transition: real vs. complex instantons — GD, Dumlu, 1004.2509, 1102.2899
- the same transition as in the Mathieu equation — Basar, GD, 1501.05671
- non-trivial quantum interference effects for general E(t)

Resurgence in QFT: Ultra-Fast Dynamics

time evolution of quantum systems with ultra-fast perturbations

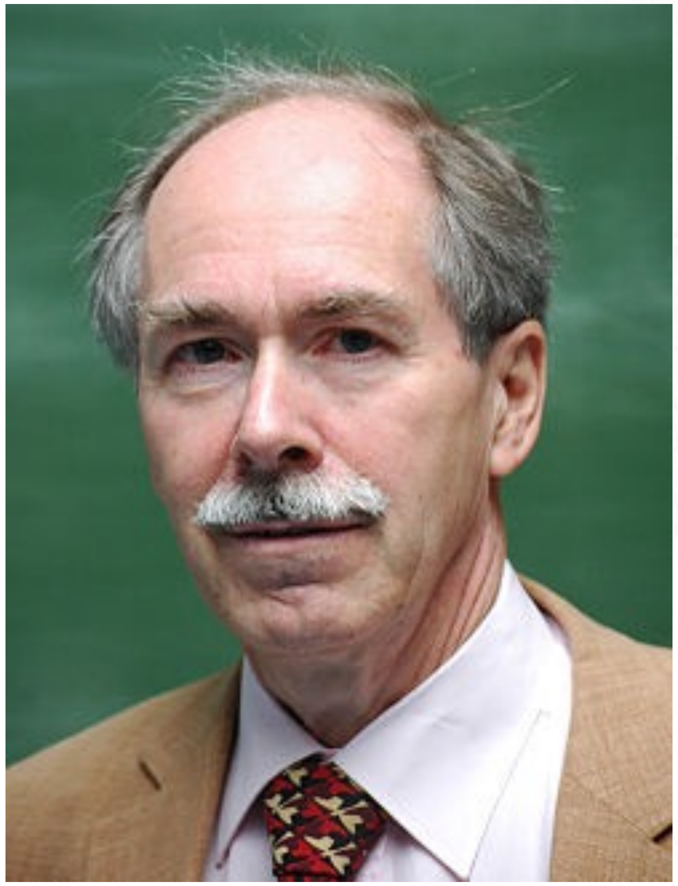


- the adiabatic expansion is divergent
- resurgence: expansion can be (Borel) resummed to a <u>universal form</u>
- novel quantum interference effects: complex saddles
- applications in AMO and CM physics and in QFT

Resurgence in Asymptotically Free Quantum Field Theory

CAN WE MAKE SENSE OUT OF "QUANTUM CHROMODYNAMICS"?

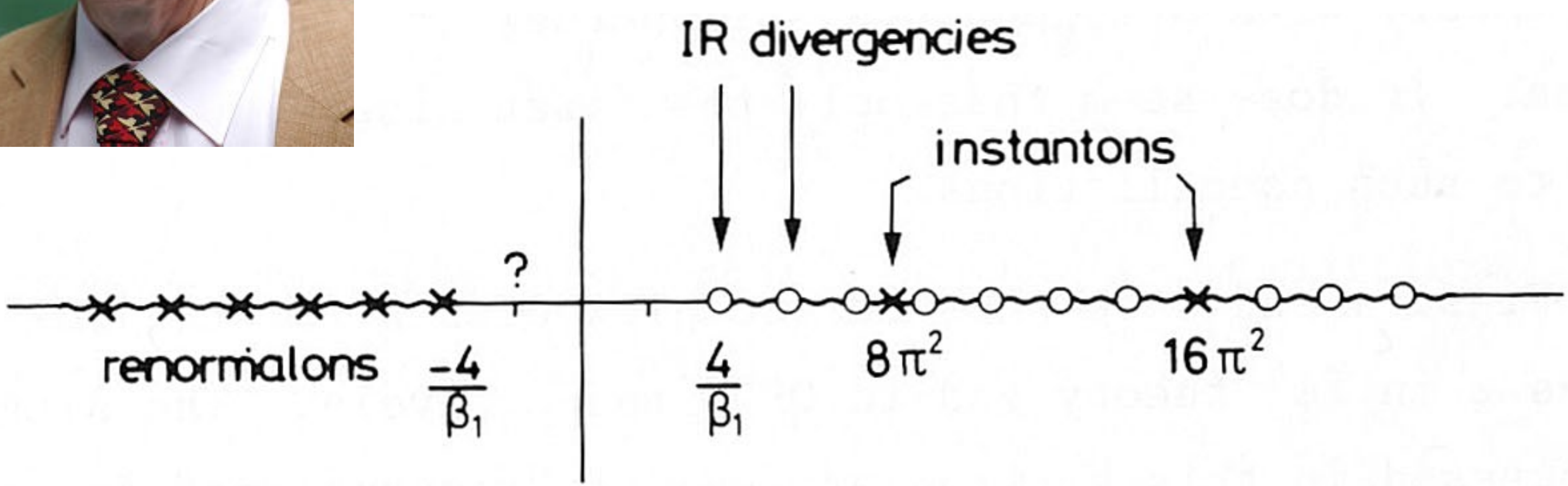
1979



G. 't Hooft

Institute for Theoretical Physics

University of Utrecht, Netherlands



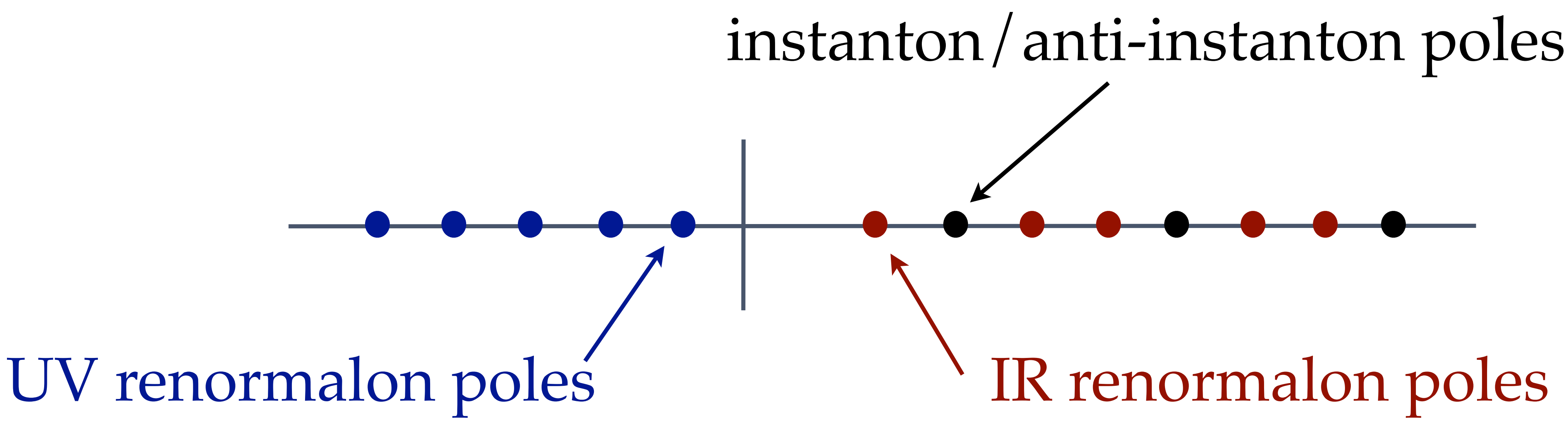
"infrared renormalon puzzle": the BZJ cancelation appears to fail …

Resurgence in Quantum Field Theory

infrared renormalon puzzle of asymptotically free QFT

perturbation theory + Borel: $\longrightarrow$ $+i\,\exp\left[-\frac{2\,S_I}{g^2\,\beta_1}\right]$

non-perturbative instantons : $\longrightarrow$ $-i\,\exp\left[-\frac{2\,S_I}{g^2}\right]$



Resurgence in Quantum Field Theory

infrared renormalon puzzle of asymptotically free QFT

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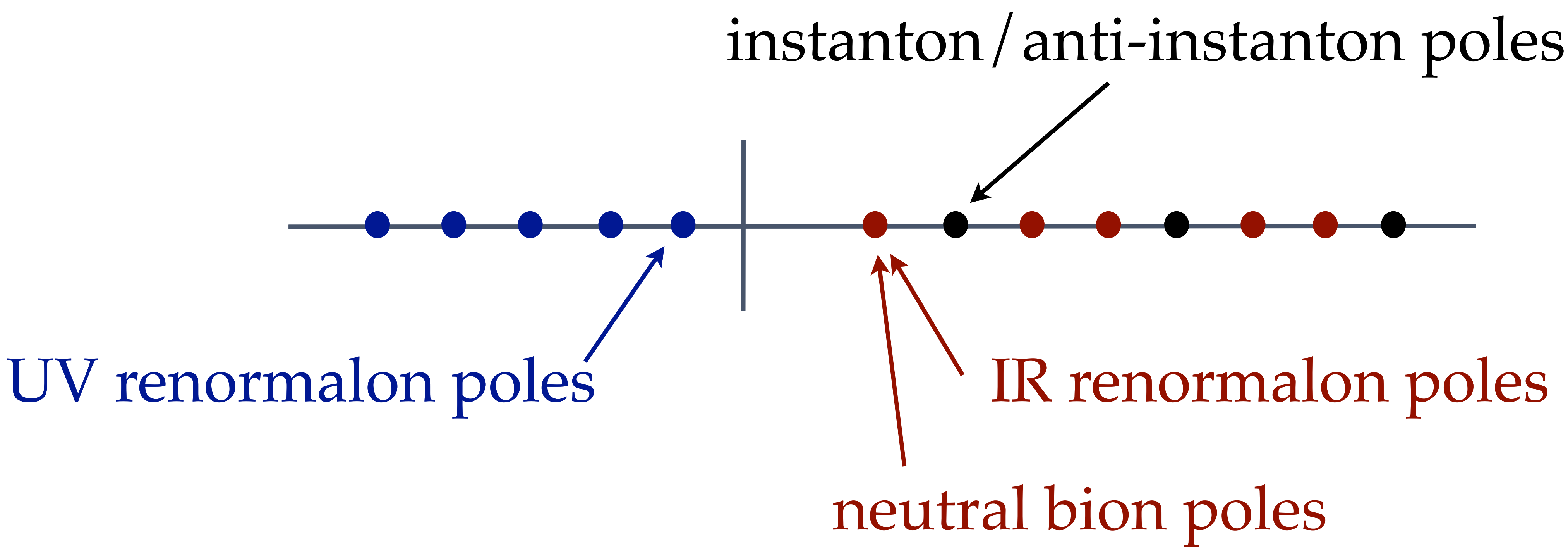


$$\longrightarrow \quad -i\, \exp\left[-\frac{2\, S_I}{g^2\, \beta_1}\right]$$

new non-perturbative objects ("neutral bions") lead to
Bogomolny/Zinn-Justin style resurgent cancelation

GD/Unsal, 1210.2423

Analytic Continuation of Path Integrals: "Lefschetz Thimbles"

$$Z(\hbar) = \int \mathcal{D}A \, \exp\left(\frac{i}{\hbar} S[A]\right) \stackrel{?}{=} \sum_{\text{thimble}} \mathcal{N}_{\text{th}} \, e^{i\,\phi_{\text{th}}} \int_{\text{th}} \mathcal{D}A \, \times (\mathcal{J}_{\text{th}}) \times \exp\left(\mathcal{R}e\left[\frac{i}{\hbar} S[A]\right]\right)$$

Lefschetz thimble = "functional steepest descents contour"

on a thimble, the path integral becomes
well-defined and computable !

complexified gradient flow:

$$\frac{\partial}{\partial \tau} A(x;\tau) = -\overline{\frac{\delta S}{\delta A(x;\tau)}}$$

Analytic Continuation of Path Integrals: "Lefschetz Thimbles"

CRISTOFORETTI *et al.* (2013)

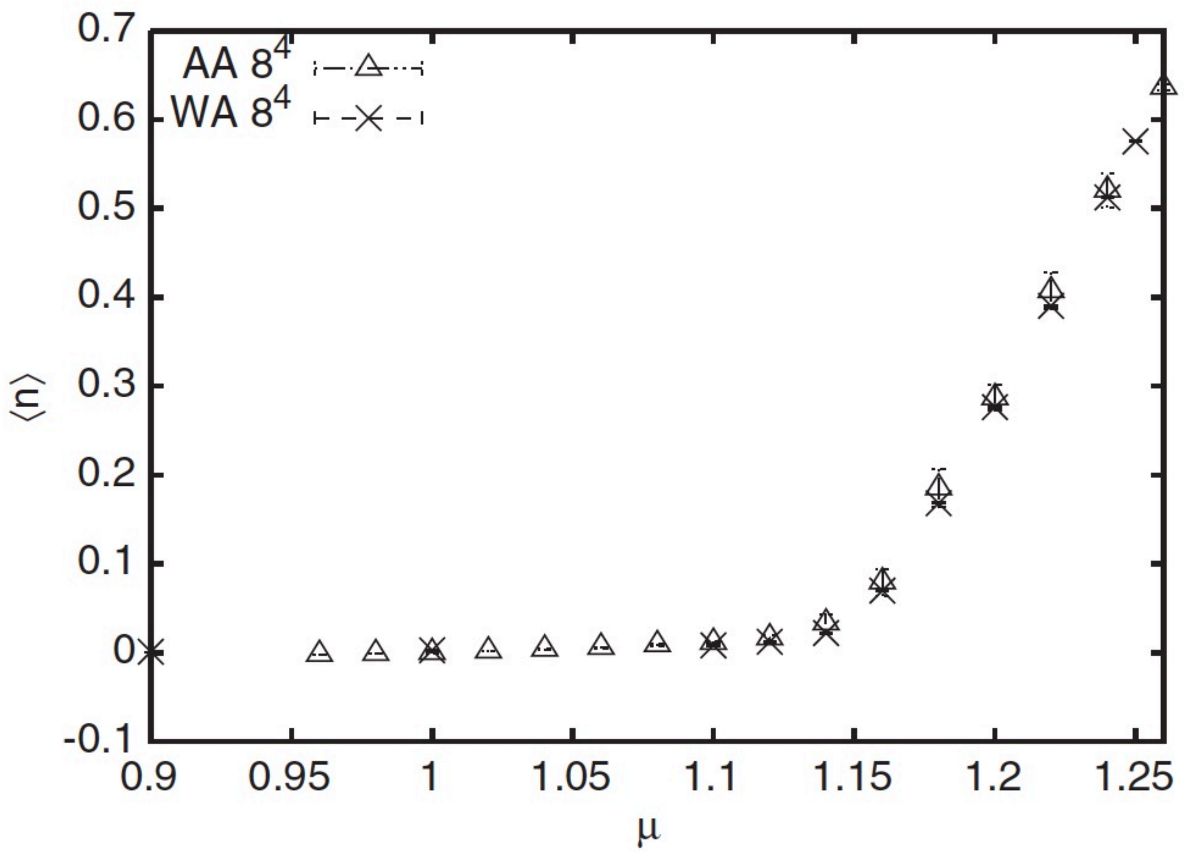
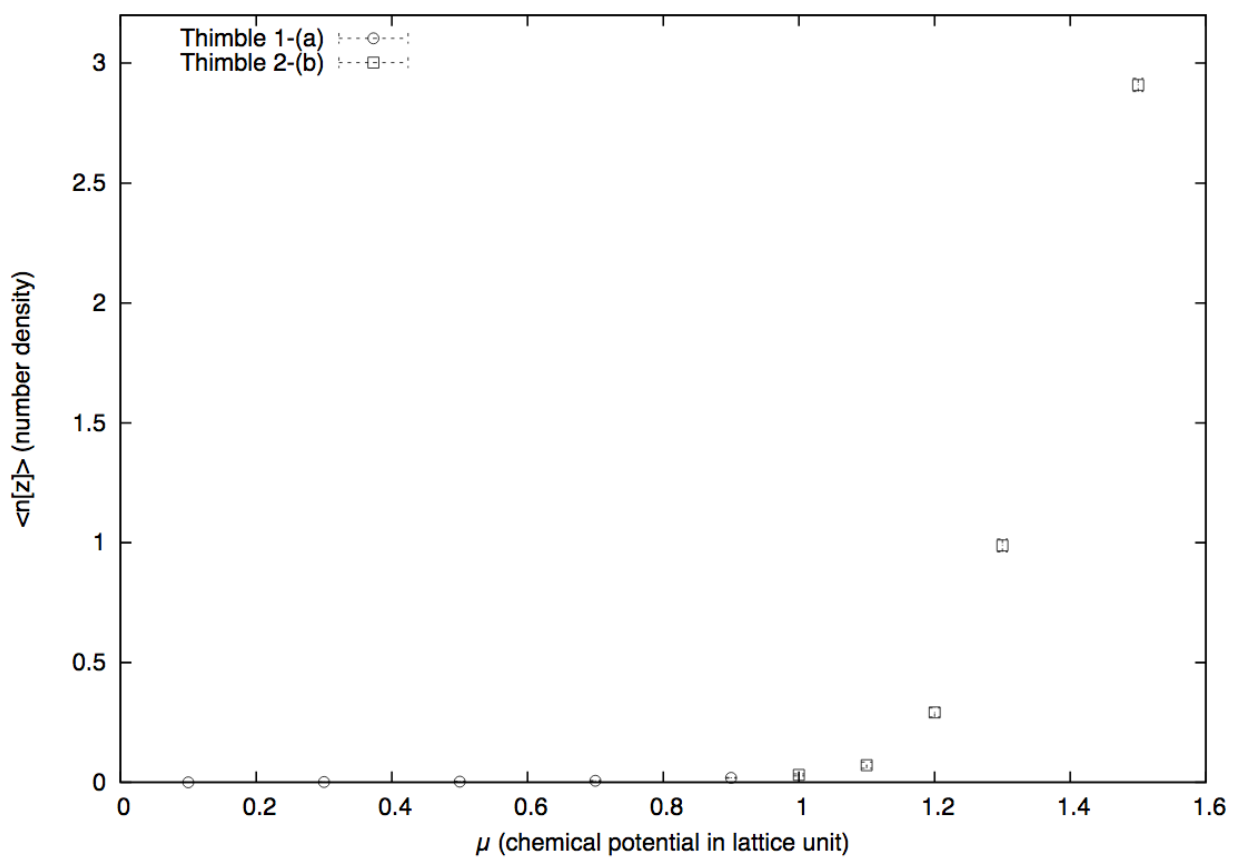


FIG. 3. Comparison of the average density $\langle n \rangle$ obtained with the worm algorithm (WA) [22] with the Aurora algorithm (AA)

Fujii et al (2013)

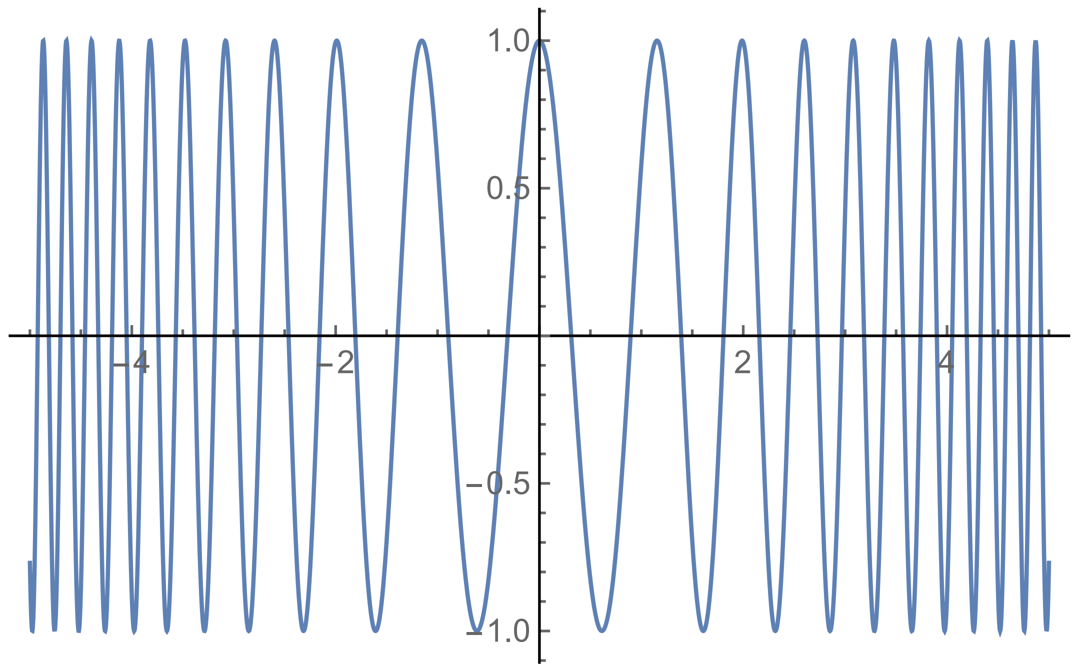


- 4d relativistic Bose gas: complex scalar field theory
- Monte Carlo on thimble softens the sign problem
- results comparable to "worm algorithm"

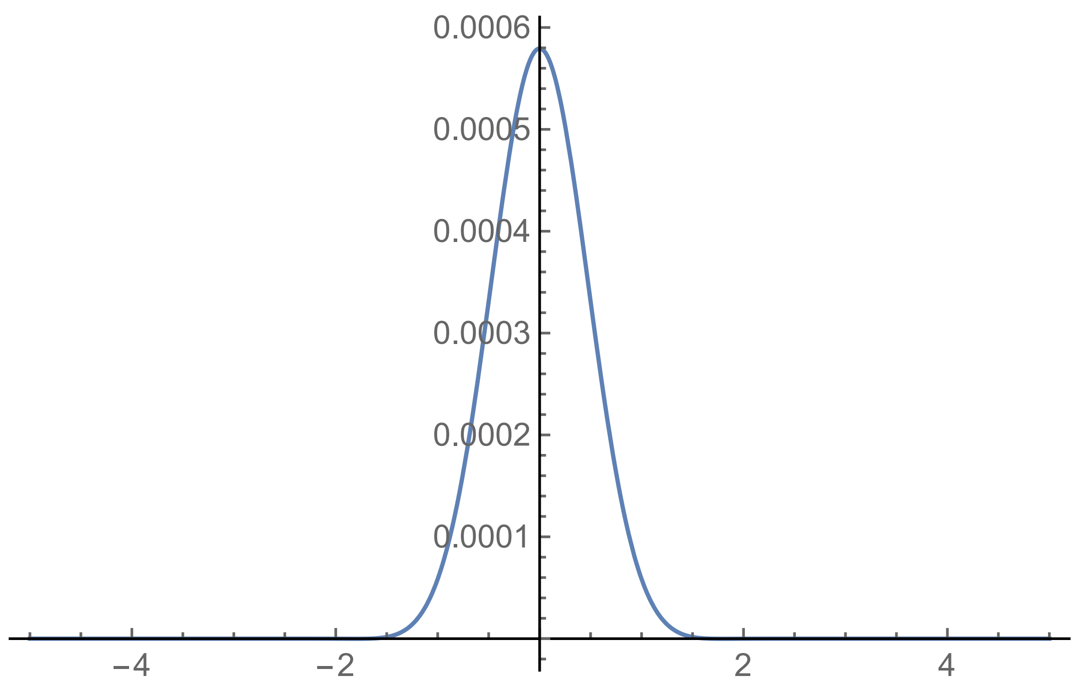
Generalized Thimble Method

Alexandru, Basar, Bedaque et al 2016

idea: flow to an approximate Lefschetz thimble



exact steepest descents contour



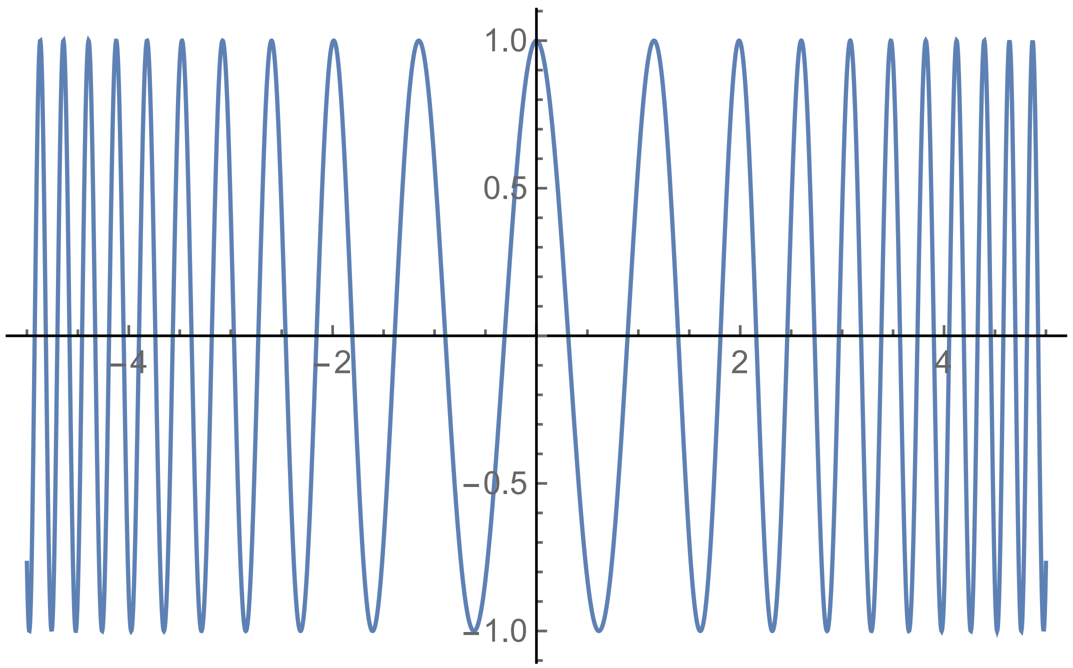
$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\left(\frac{1}{3}x^3 + z\,x\right)}dx$$

Generalized Thimble Method

Alexandru, Basar, Bedaque et al 2016

idea: flow to an approximate Lefschetz thimble

$$\mathrm{Ai}(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\left(\frac{1}{3}x^3 + z\,x\right)}\,dx$$



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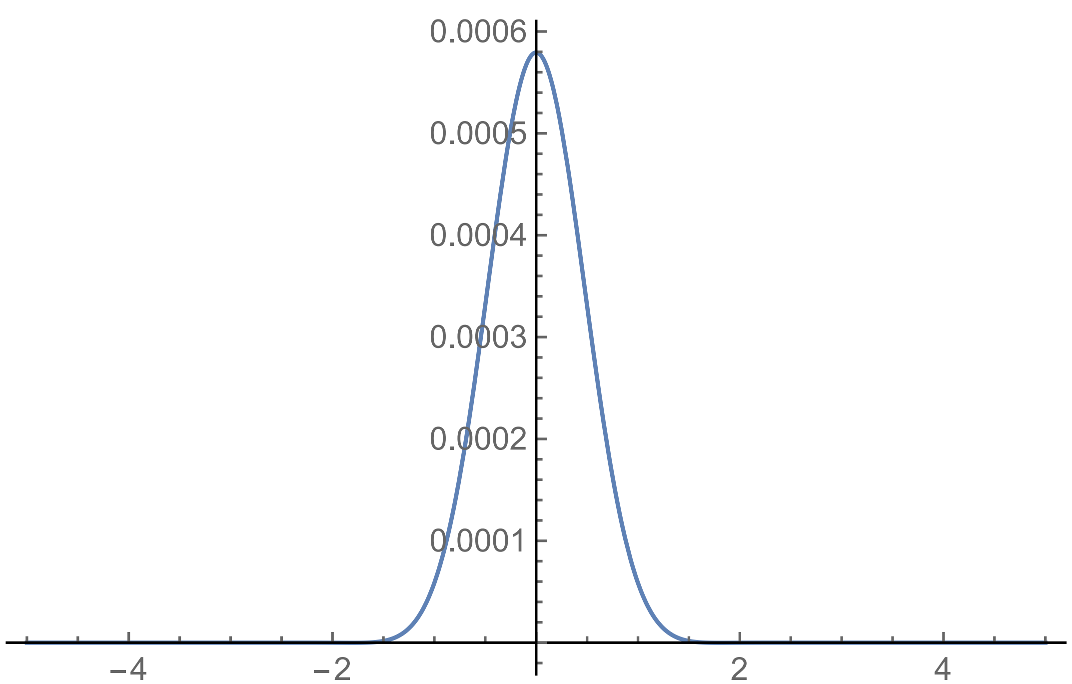
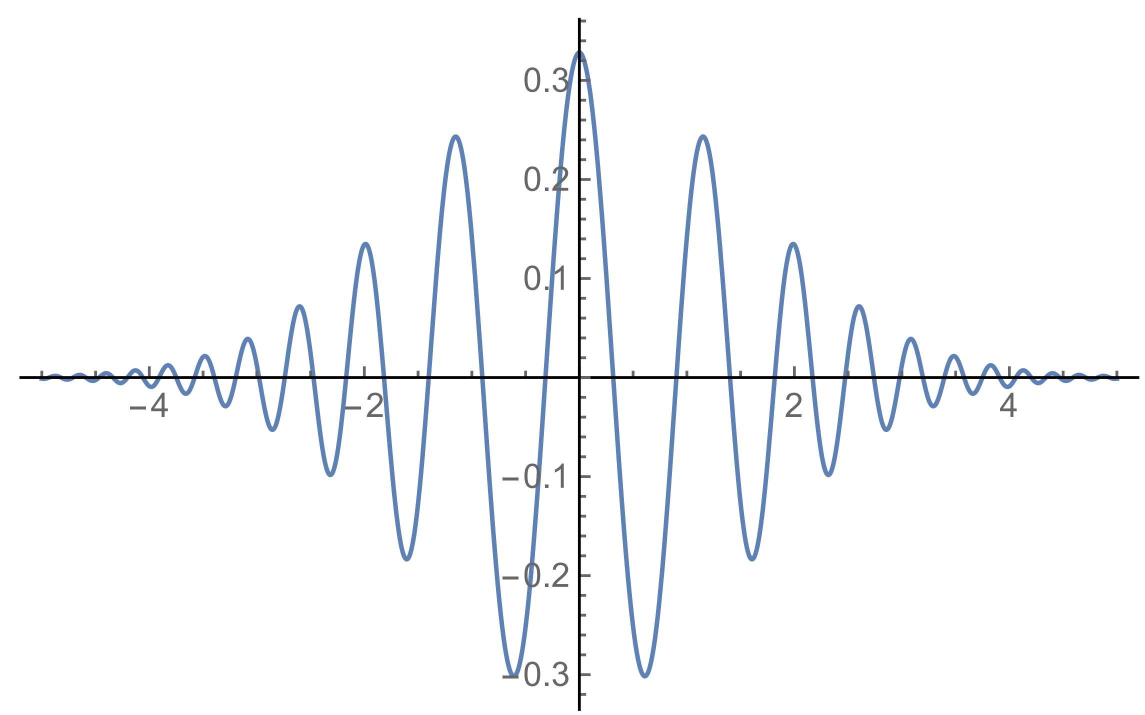
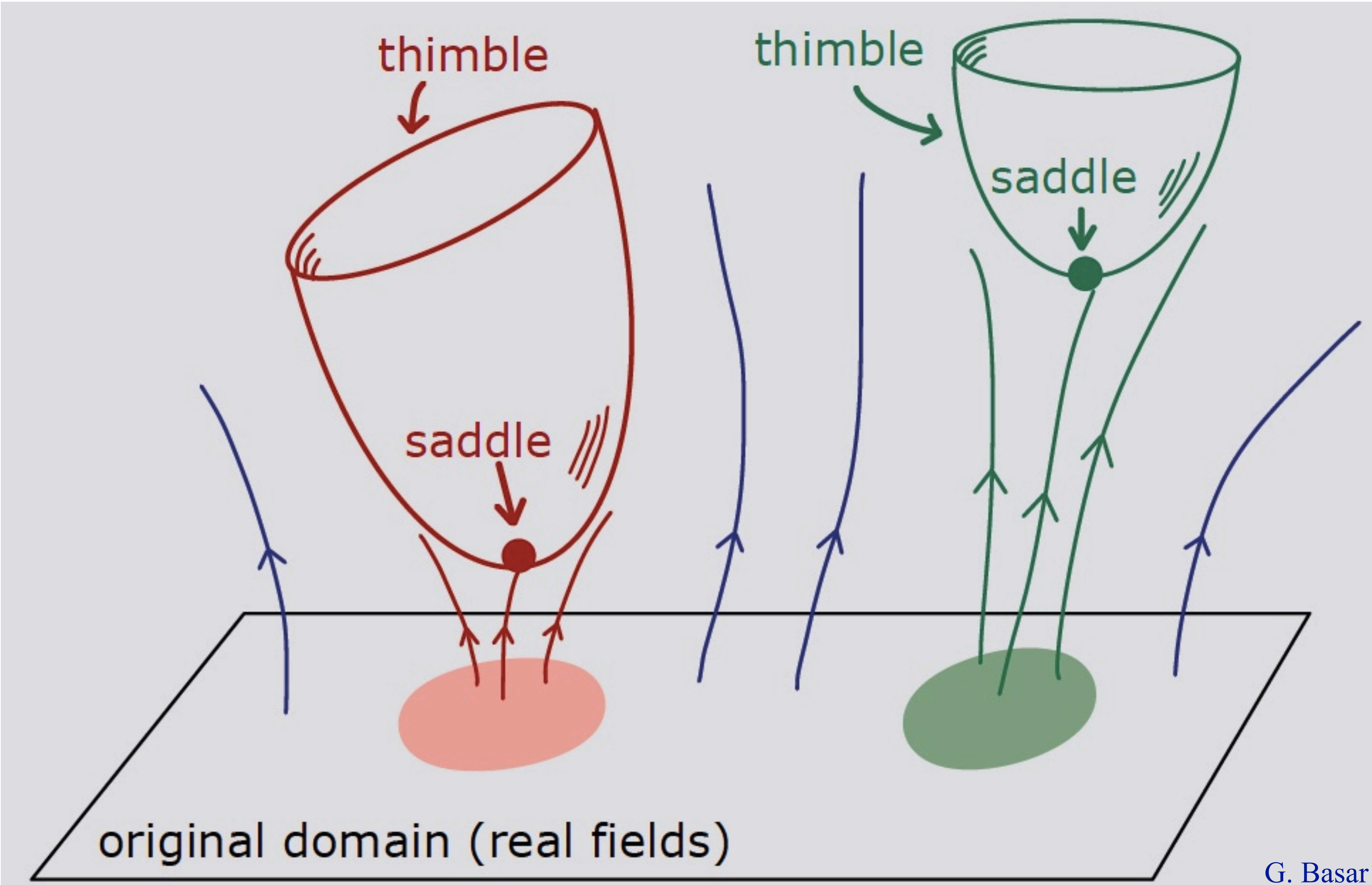


approximate steepest descents contour



Generalized Thimble Method



recall that thimble structure can change as parameters change

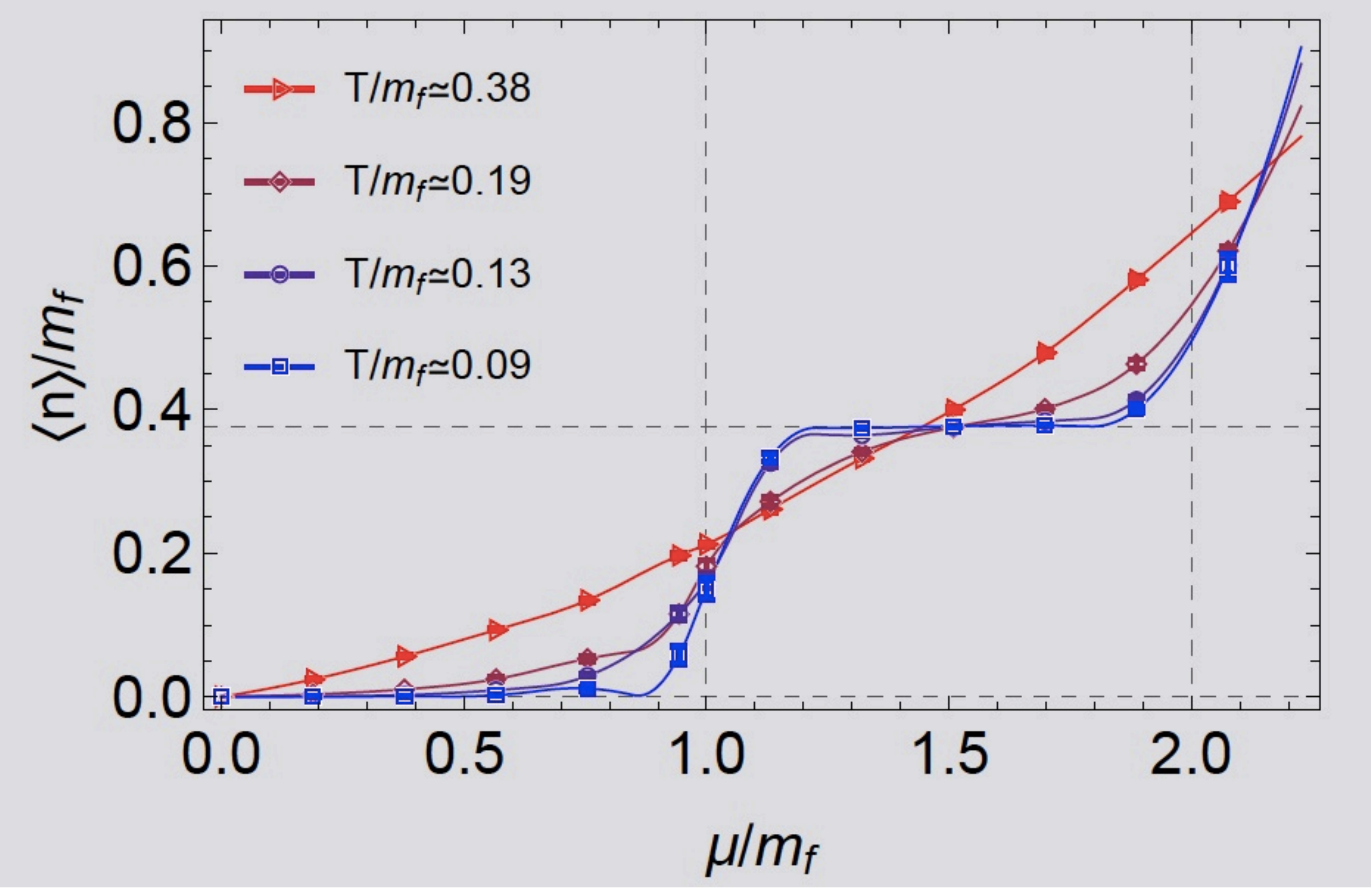
Phase Transitions in QFT: 2d Thirring Model

$$\mathcal{L} = \bar{\psi}^a \left(\gamma_\nu \partial_\nu + m + \mu\, \gamma_0\right) \psi^a + \frac{g^2}{2N_f} \left(\bar{\psi}^a \gamma_\nu \psi^a\right) \left(\bar{\psi}^b \gamma_\nu \psi^b\right)$$

- chain of interacting fermions: asymptotically free
- prototype for dense quark matter
- sign problem at nonzero density
- cousin of Hubbard model

Monte Carlo thimble computation

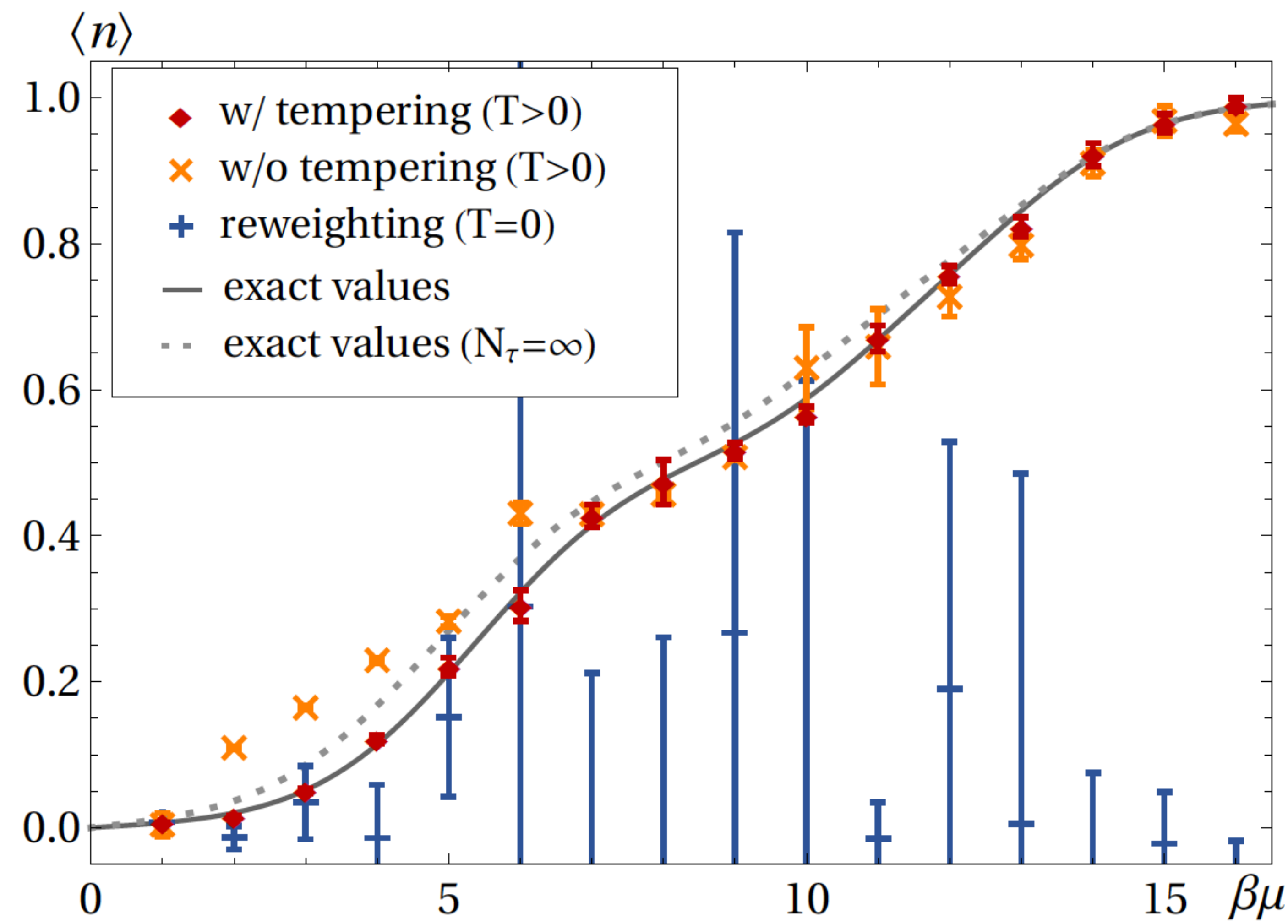
(Alexandru et al, 2016)



Tempered Lefschetz Thimble Method

(Fukuma et al, 2017, 2019,…)

- probe all relevant thimbles ???
- sign problem vs. ergodicity
- coupling → dynamical variable
- parallelized tempering
- e.g. 2d Hubbard model
- probes multiple thimbles



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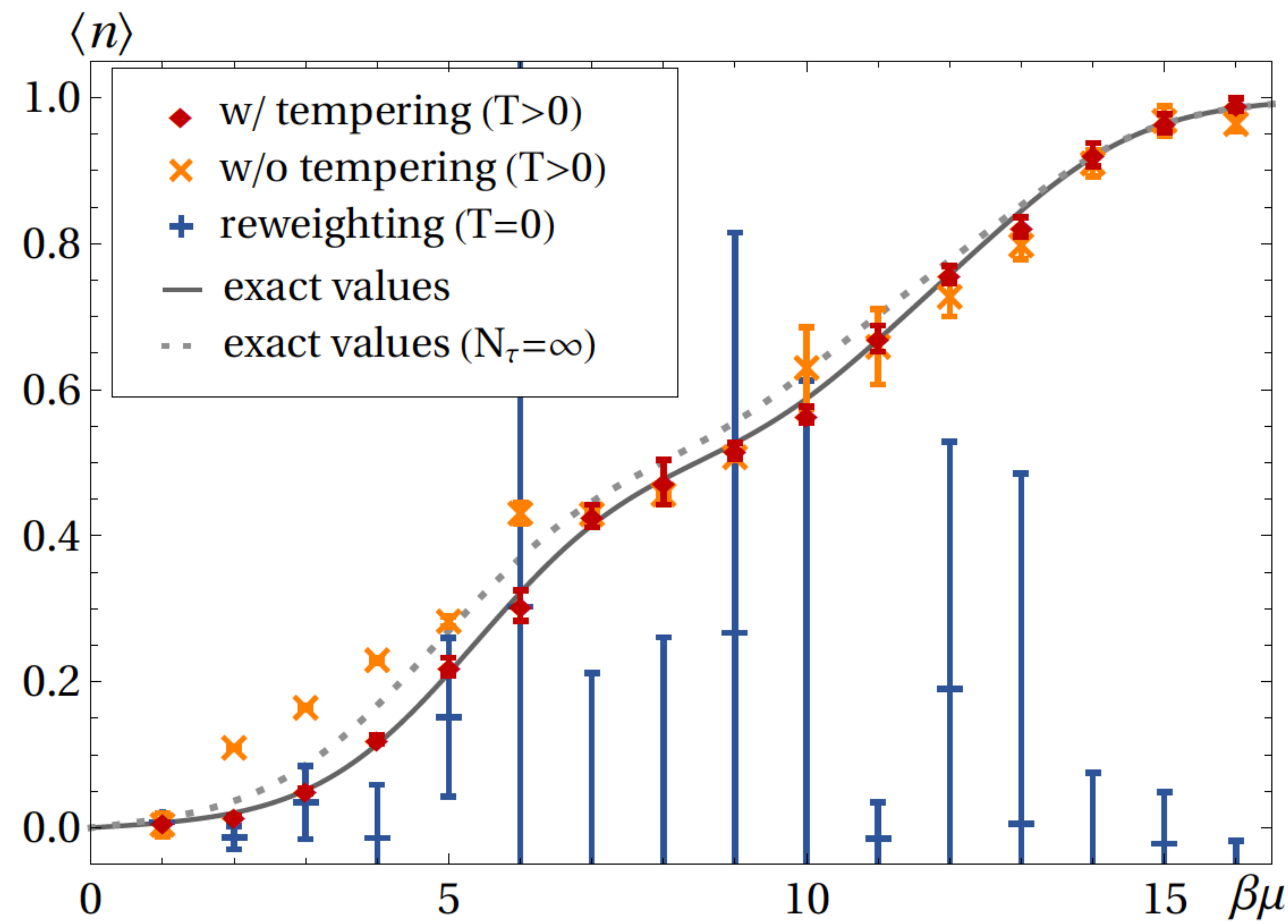
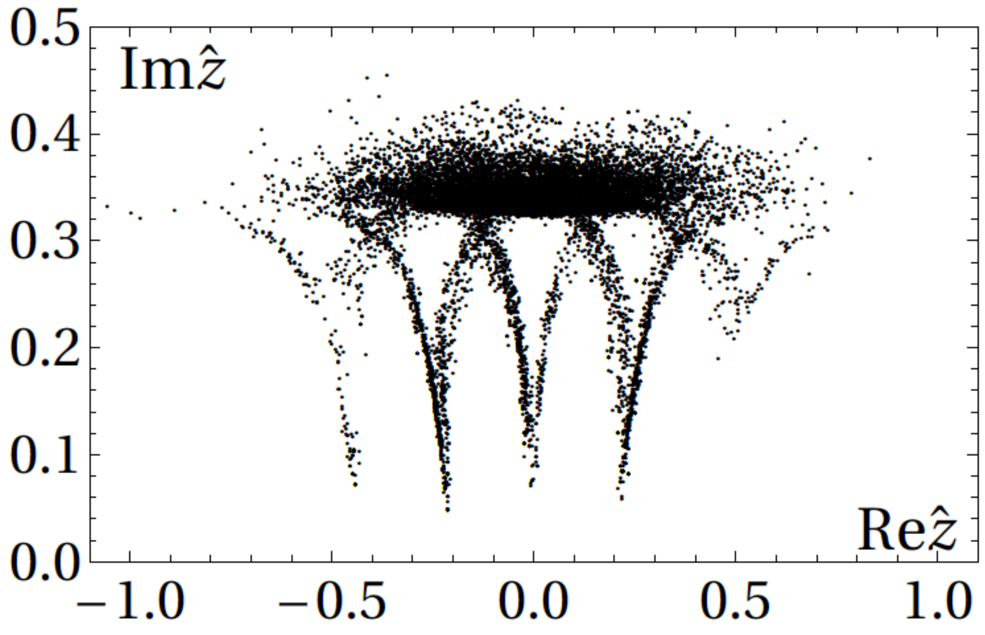
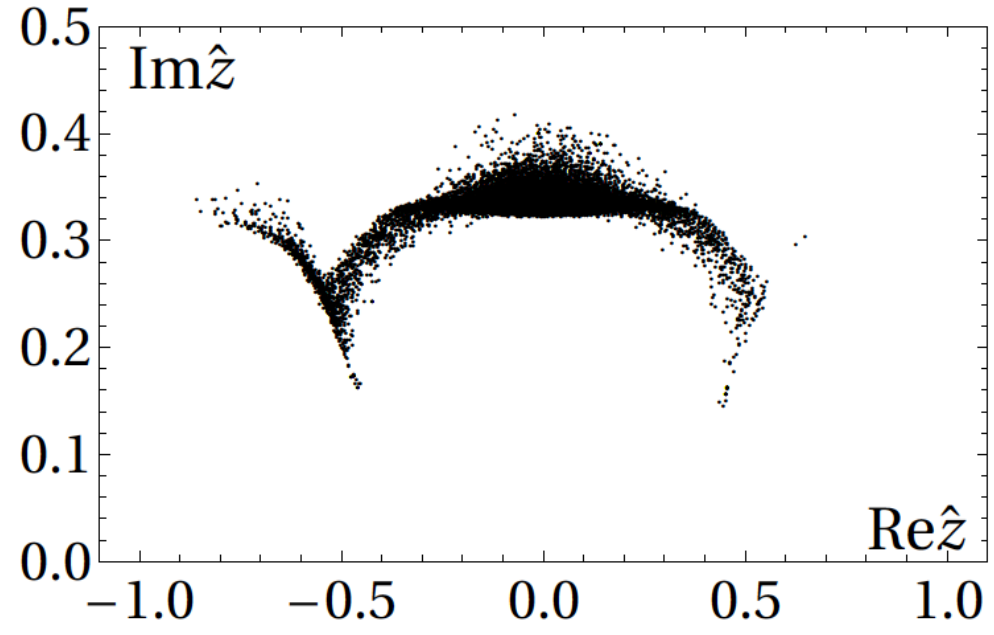




with tempering

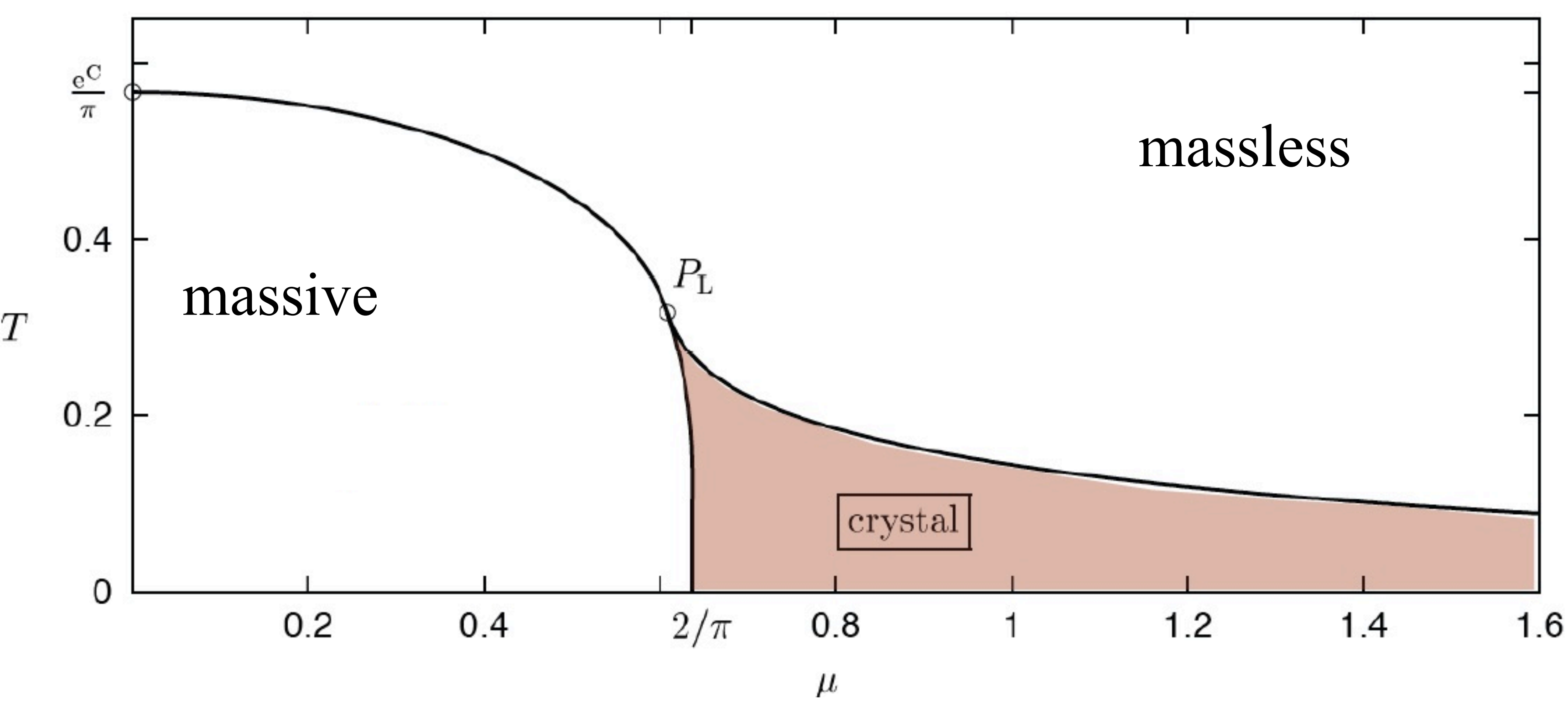


without tempering

Phase Transitions in 2d Gross-Neveu Model

$$\mathcal{L}_{\rm Gross-Neveu} = \bar{\psi}_a i\partial\!\!\!/ \psi_a + \frac{g^2}{2}\left(\bar{\psi}_a \psi_a\right)^2$$

- asymptotically free; dynamical mass; chiral symmetry; model for QCD
- large N_f chiral symmetry breaking phase transition
- physics = (relativistic) Peierls dimerization instability in 1+1 dim.



chiral symmetry breaking condensate

$$\sigma(x;T,\mu) \equiv \langle\bar{\psi}\psi\rangle(x;T,\mu)$$

develops crystalline phases !

(Thies et al)

saddles solve inhomogeneous gap equation

$$\sigma(x;T,\mu) = \frac{\delta}{\delta\sigma(x;T,\mu)} \ln\det\left(i\,\partial\!\!\!/ - \sigma(x;T,\mu)\right)$$

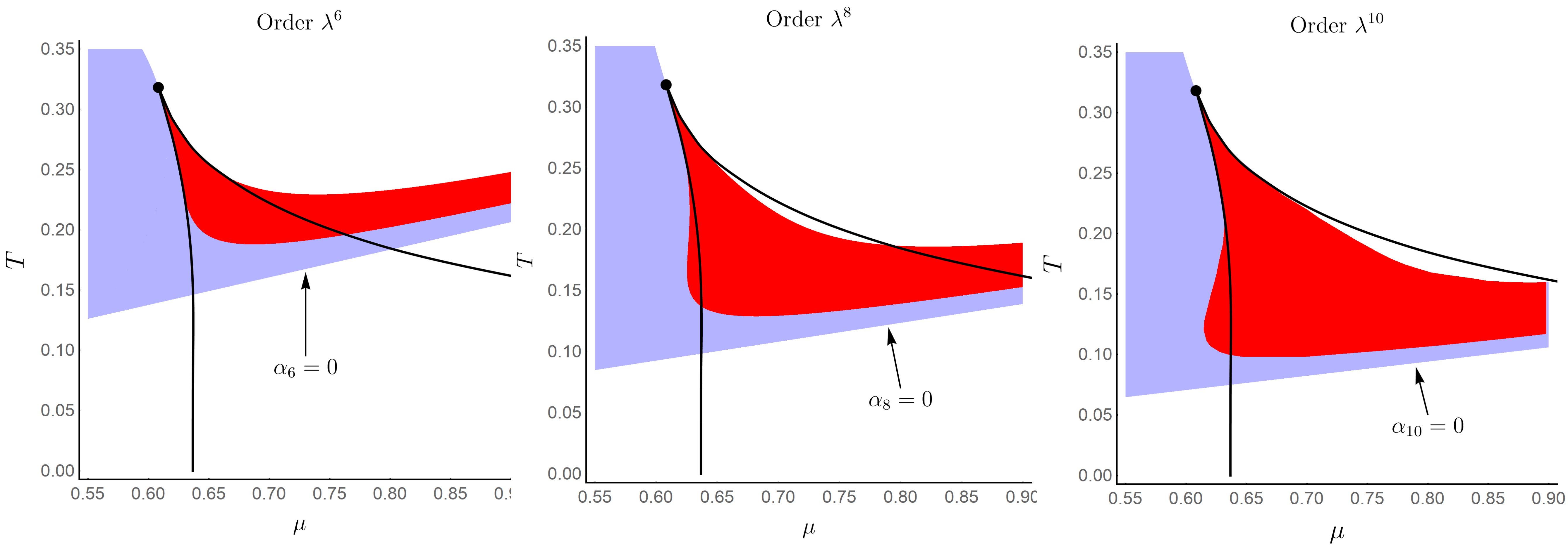
Phase Transitions in Gross-Neveu Model

- thermodynamic potential

Basar, GD, Thies, 0903.1868

$$\Psi[\sigma; T, \mu] = \sum_n \alpha_n(T,\mu) f_n[\sigma(x,T,\mu)] = \alpha_0 + \alpha_2\,\sigma^2 + \alpha_4\left(\sigma^4 + (\sigma')^2\right) + \ldots$$

- (divergent) Ginzburg-Landau expansion = mKdV hierarchy
- exact saddles are known
- successive orders of GL expansion “reveal” crystal phase



- all orders gives full crystal phase … but T=0 critical point is difficult

Phase Transitions in Gross-Neveu Model

- density expansion has non-perturbative terms: “trans-series”

- high-density expansion at T=0: (convergent)

$$\mathcal{E}(\rho) \sim \frac{\pi}{2}\,\rho^2\left(1-\frac{1}{32(\pi\rho)^4}+\frac{3}{8192(\pi\rho)^8}-\ldots\right)$$

- low-density expansion at T=0: (non-perturbative trans-series)

$$\mathcal{E}(\rho) \sim -\frac{1}{4\pi}+\frac{2\rho}{\pi}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{e^{-k/\rho}}{\rho^{k-2}}\mathcal{F}_{k-1}(\rho)$$

- T=0 quantum phase transition

$$\mu_{\rm critical}=\frac{2}{\pi} \quad \leftrightarrow \quad \rho=0$$

Resurgence and Large N Phase Transitions in Matrix Models

3rd order phase transition in Gross-Witten-Wadia unitary matrix model

$$Z(t, N) = \int_{U(N)} DU \, \exp\left[\frac{N}{t} \mathrm{tr}\left(U + U^\dagger\right)\right]$$

Gross-Witten, 1980
Wadia, 1980
Marino, 2008

Z depends on two parameters: 't Hooft coupling t, and matrix size N

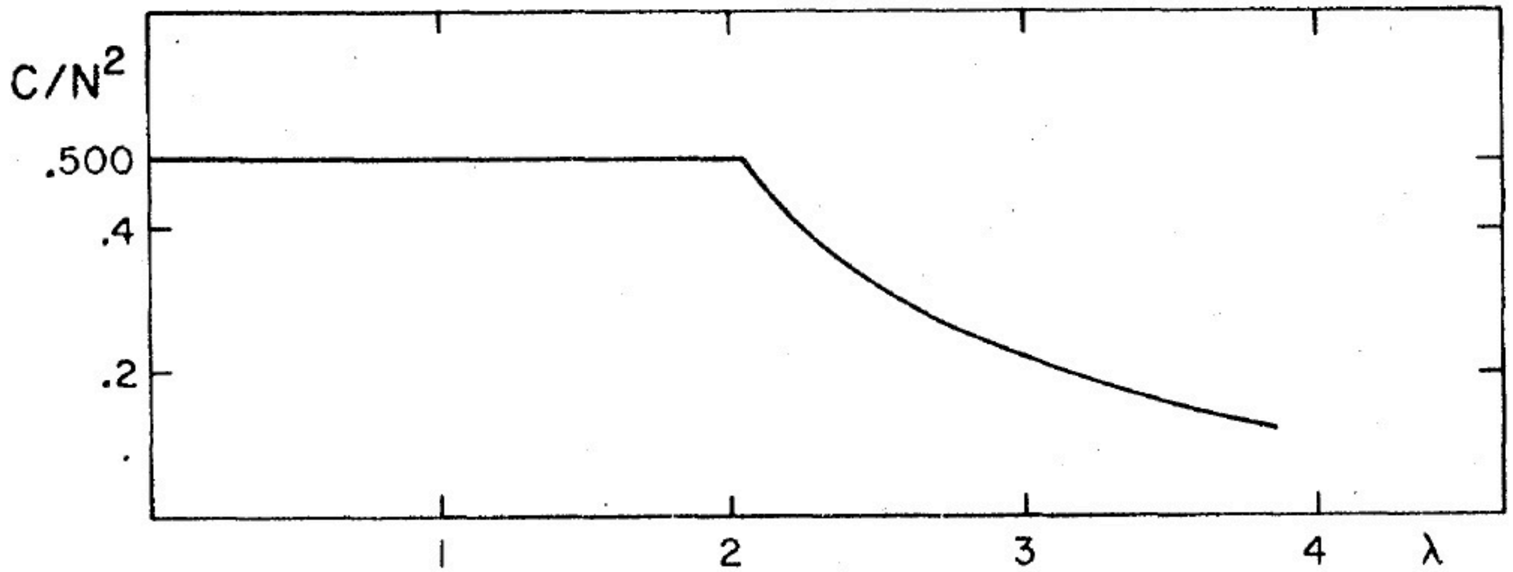


FIG. 2. The specific heat per degree of freedom, C/N^2, as a function of λ (temperature).

$$Z(t, N) = \det\left[I_{j-k}\left(\frac{N}{t}\right)\right]_{j,k=1,\ldots N}$$

phase transition in the ``thermodynamic'' large N limit

Resurgence in Matrix Models at Large N

"order parameter" $\Delta(t, N) \equiv \langle \det U \rangle$ satisfies a Painleve III equation

Ahmed & GD, 1710.01812

$$t^2 \Delta'' + t\Delta' + \frac{N^2 \Delta}{t^2} \left(1 - \Delta^2\right) = \frac{\Delta}{1 - \Delta^2} \left(N^2 - t^2 \left(\Delta'\right)^2\right)$$

N appears only as a parameter: perfect for large N asymptotics

$$\Delta(t, N) \sim \sum_n \frac{a_n(t)}{N^n} + e^{-N\, S(t)} \sum_n \frac{b_n(t)}{N^n} + e^{-2N\, S(t)} \sum_n \frac{c_n(t)}{N^n} + \ldots$$

large N instanton contributions: generated from ODE

$$\text{e.g.} \quad a_0(t) = \sqrt{1 - t}$$

all physical observables inherit the large N trans-series structure

Resurgence in Matrix Models at Large N

ODE $\Rightarrow$ large N weak coupling trans-series:

$$\Delta(t,N) \sim \sqrt{1-t}\sum_{n=0}^{\infty}\frac{d_n^{(0)}(t)}{N^{2n}} - \frac{i\,\sigma_{\rm weak}}{2\sqrt{2\pi N}}\,\frac{t\,e^{-N S_{\rm weak}(t)}}{(1-t)^{1/4}}\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} + \ldots$$

weak coupling large N action:

$$S_{\rm weak}(t) = \frac{2\sqrt{1-t}}{t} - 2\,\mathrm{arctanh}\left(\sqrt{1-t}\right)$$

Resurgence in Matrix Models at Large N

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“one-instanton" fluctuations:

$$\sum_{n=0}^{\infty}\frac{d_n^{(1)}(t)}{N^n} = 1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\,\frac{1}{N} + \dots$$

Resurgence in Matrix Models at Large N

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resurgence: large-order growth of “perturbative coefficients”:

$$d_n^{(0)}(t) \sim \frac{-1}{\sqrt{2}(1-t)^{3/4}\pi^{3/2}}\frac{\Gamma(2n-\frac{5}{2})}{(S_{\text{weak}}(t))^{2n-\frac{5}{2}}}\left[1 + \frac{(3t^2-12t-8)}{96(1-t)^{3/2}}\frac{S_{\text{weak}}(t)}{(2n-\frac{7}{2})} + \dots\right]$$

Resurgence in Matrix Models at Large N

large N strong coupling trans-series: completely different structure

$$\Delta(t,N) \sim \frac{\sqrt{t}\, e^{-N S_{\rm strong}(t)}}{\sqrt{2\pi N}\,(t^2-1)^{1/4}} \sum_{n=0}^{\infty} \frac{U_n\left(q(t)\right)}{N^n}$$

$$+\frac{1}{4(t^2-1)}\left(\frac{\sqrt{t} e^{-N S_{\rm strong}(t)}}{\sqrt{2\pi N}\,(t^2-1)^{1/4}}\right)^3 \sum_{n=0}^{\infty} \frac{U_n^{(1)}\left(q(t)\right)}{N^n} + \dots$$

strong coupling large N action:

$$S_{\rm strong}(t) = \text{arccosh}\,(t) - \sqrt{1-\frac{1}{t^2}}$$

Resurgence in Matrix Models at Large N

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$$S_{\rm strong}(t) = \text{arccosh}\,(t) - \sqrt{1-\frac{1}{t^2}}$$

resurgence: large-order growth of “perturbative coefficients”:

$$U_n\,(q(t)) \sim \frac{1}{2\pi}\frac{(-1)^n\,(n-1)!}{(2S_{\rm strong}(t))^n}\left(1 + U_1\,(q(t))\,\frac{(2S_{\rm strong}(t))}{(n-1)} + U_2\,(q(t))\,\frac{(2S_{\rm strong}(t))^2}{(n-1)(n-2)} + \dots\right)$$

Large N Transmutation of Transseries

weak-coupling trans-series changes its form across the phase transition into the strong-coupling phase

immediate vicinity of $t=1$ is described by Painleve II equation ("double-scaling limit")

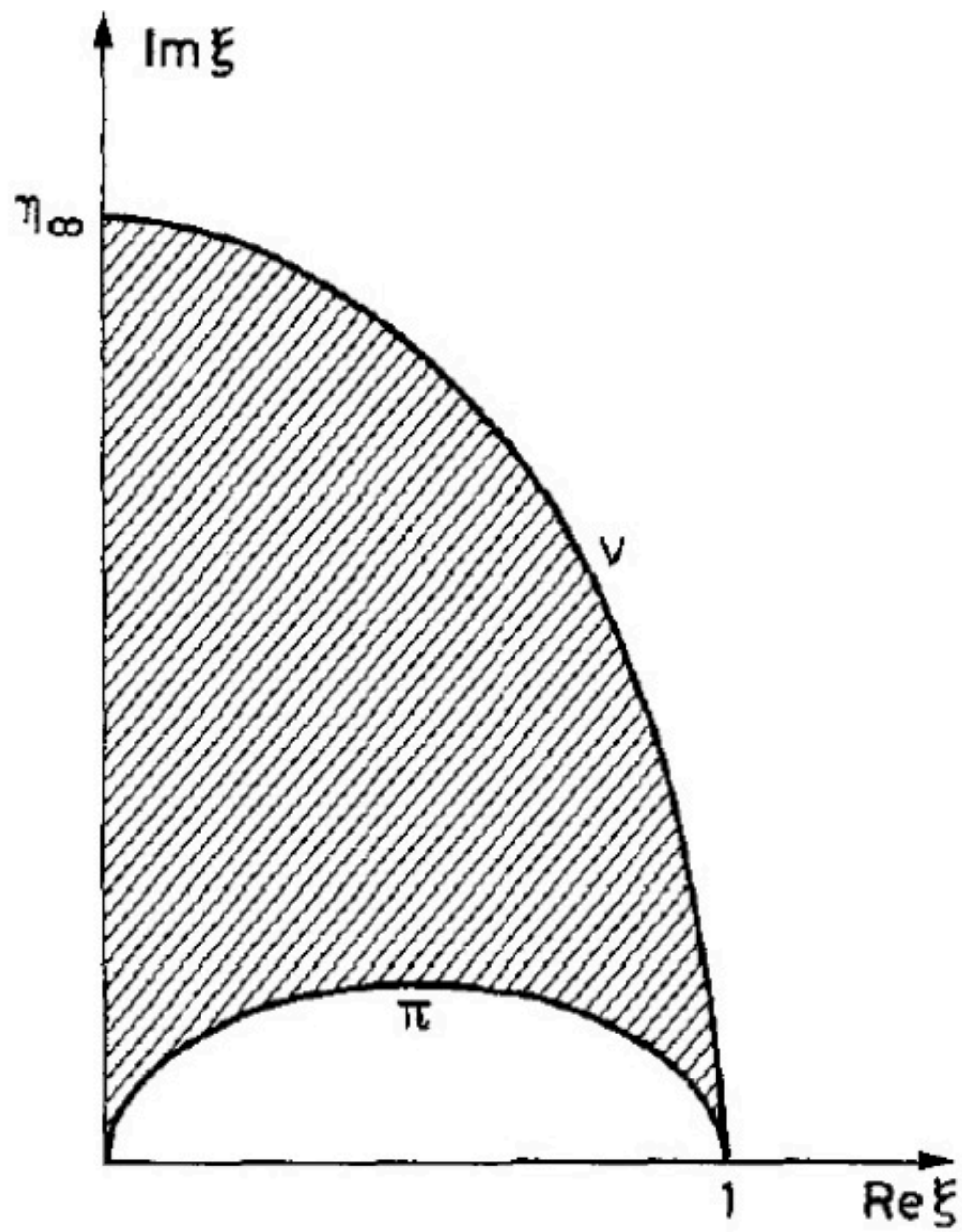
physics = instanton condensation

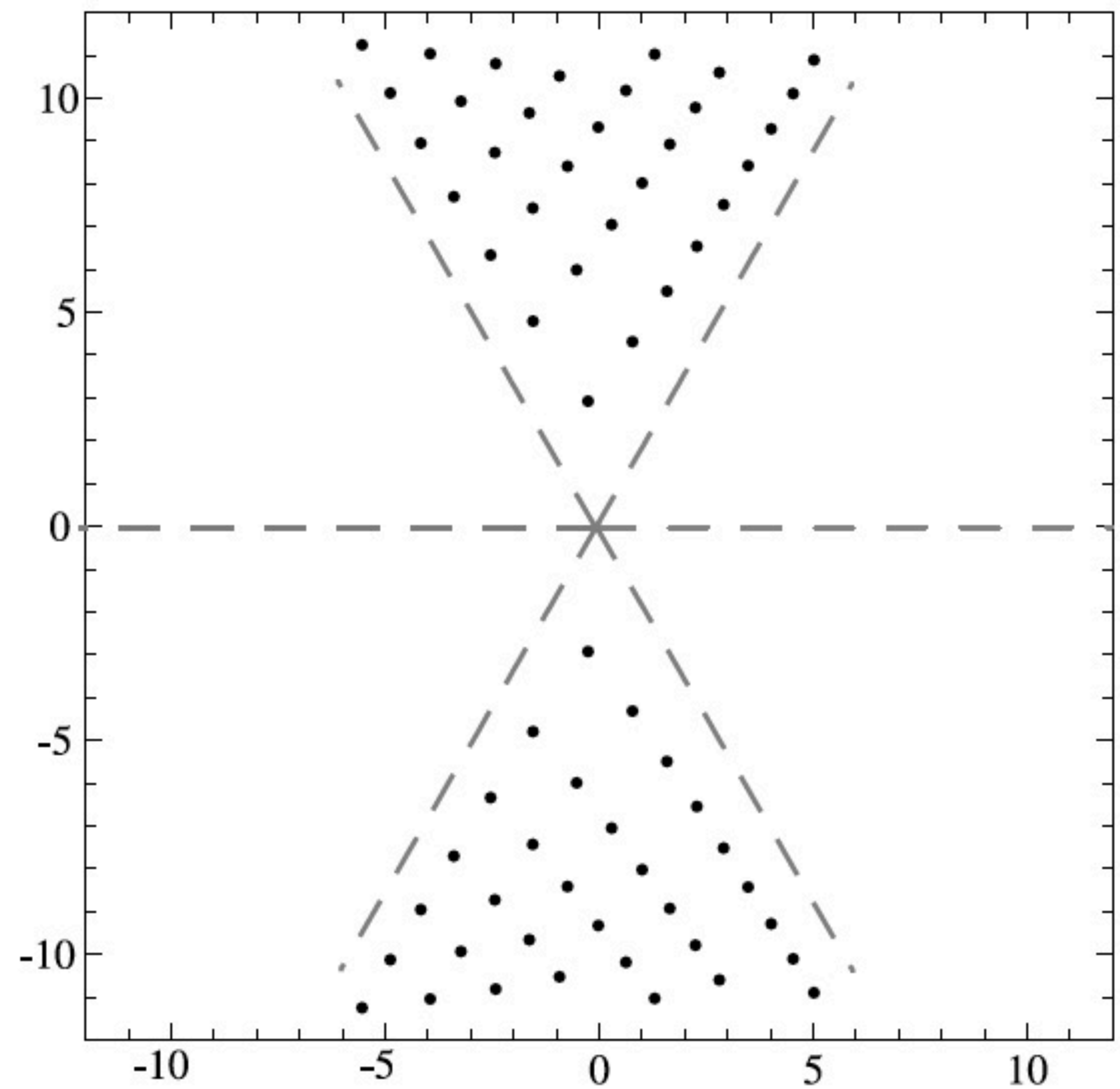
physics = Stokes transition between real and complex instantons

universal transition: cf. Mathieu & Schwinger effect examples

Lee-Yang view of Large N Phase Transitions in Matrix Models

Lee-Yang: complex zeros of Z(t, N) pinch the real t axis at the phase transition, in the thermodynamic (large N) limit





complex parameters can indicate phase transitions

2 dim Yang-Mills: Douglas-Kazakov Large N Phase Transition

e.g., 2d Yang-Mills on sphere

"spectral sum" for partition function:

$$Z(a,N) = \sum_R (\dim R)^2 \, e^{-\frac{a}{2N} C_2(R)}$$

large N phase transition at critical area

"saddle sum" for partition function:

$$Z(a,N) = \sum_{\vec{n}} \mathcal{F}(\vec{n}) \, e^{-\frac{2\pi^2 N}{a} \, \vec{n}^2}$$

phase transition = change of saddles

phase transition = transmutation of trans-series

dual descriptions: generalized Poisson duality

2d Lattice Ising Model

paradigm of phase transitions

Kramers-Wannier duality

$$\frac{Z(\beta J)}{\sinh^{N/2}(2\beta J)} = \frac{Z(\beta \tilde{J})}{\sinh^{N/2}(2\beta \tilde{J})} \qquad , \qquad \tanh(\beta J) \equiv e^{-2\beta \tilde{J}}$$

phase transition when $k \equiv \sinh^2(2\beta J) = 1$

expansions about T=0 and T=infinity are both convergent

2d Lattice Ising Model

paradigm of phase transitions

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phase transition when $\quad k \equiv \sinh^2(2\beta J) = 1$

expansions about T=0 and T=infinity are both convergent

$$-\beta\,\mathcal{F}(k) + \frac{1}{4}\ln k \sim \frac{1}{4}\ln k - \frac{k}{4} + \frac{k^2}{32} - \frac{k^3}{48} + \frac{9k^4}{1024} + \cdots$$

$$\sim \frac{1}{4}\ln\frac{1}{k} - \frac{1}{4k} + \frac{1}{32k^2} - \frac{1}{48k^3} + \frac{9}{1024k^4} - \cdots$$

2d Lattice Ising Model

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$$\sim \frac{1}{4}\ln\frac{1}{k} - \frac{1}{4k} + \frac{1}{32k^2} - \frac{1}{48k^3} + \frac{9}{1024k^4} - \cdots$$

resurgence: logarithmic behavior at critical T

Resurgence in 2d Lattice Ising Model

diagonal correlation functions: $C(s,N)=\langle\sigma_{0,0}\,\sigma_{N,N}\rangle(s)$

C(s, N) = tau function for Painleve VI equation (Jimbo, Miwa)

C(s, N) has a trans-series expansion: convergent about T=0, T= ∞

scaling limit: PVI → PIII as $N\to\infty$ & $T\to T_c$ (McCoy et al)

convergent conformal block expansions at low T and high T: (Lisovyy et al, 2012, 2013 …)

$$\tau(s)\sim\sum_{n=-\infty}^{\infty}\rho^n\,C(\vec{\theta},\sigma+n)\,\mathcal{B}(\vec{\theta},\sigma+n;s)$$

$$\mathcal{B}(\vec{\theta},\sigma;s)\propto s^{\sigma^2}\sum_{\lambda,\mu\in\mathcal{Y}}\mathcal{B}_{\lambda,\mu}(\vec{\theta},\sigma)s^{|\lambda|+|\mu|}$$

resurgence also for convergent expansions ! GD, 1901.02076

Resurgent Extrapolation

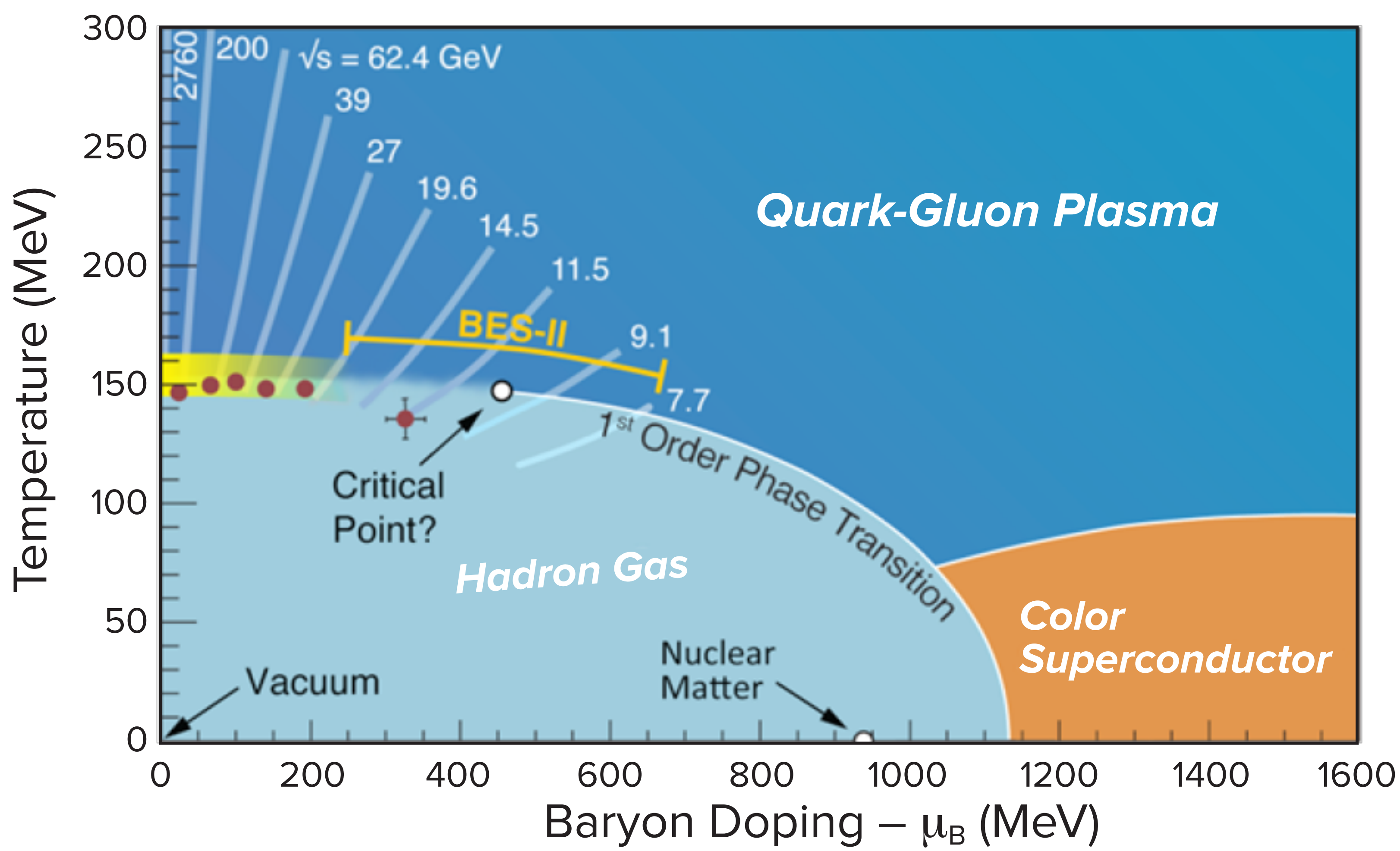
Costin, GD: 1904.11593, 2003.07451
2009.01962

- often, asymptotics is the ONLY thing we can do

- question: how much global information can be decoded from a FINITE number of perturbative coefficients ?

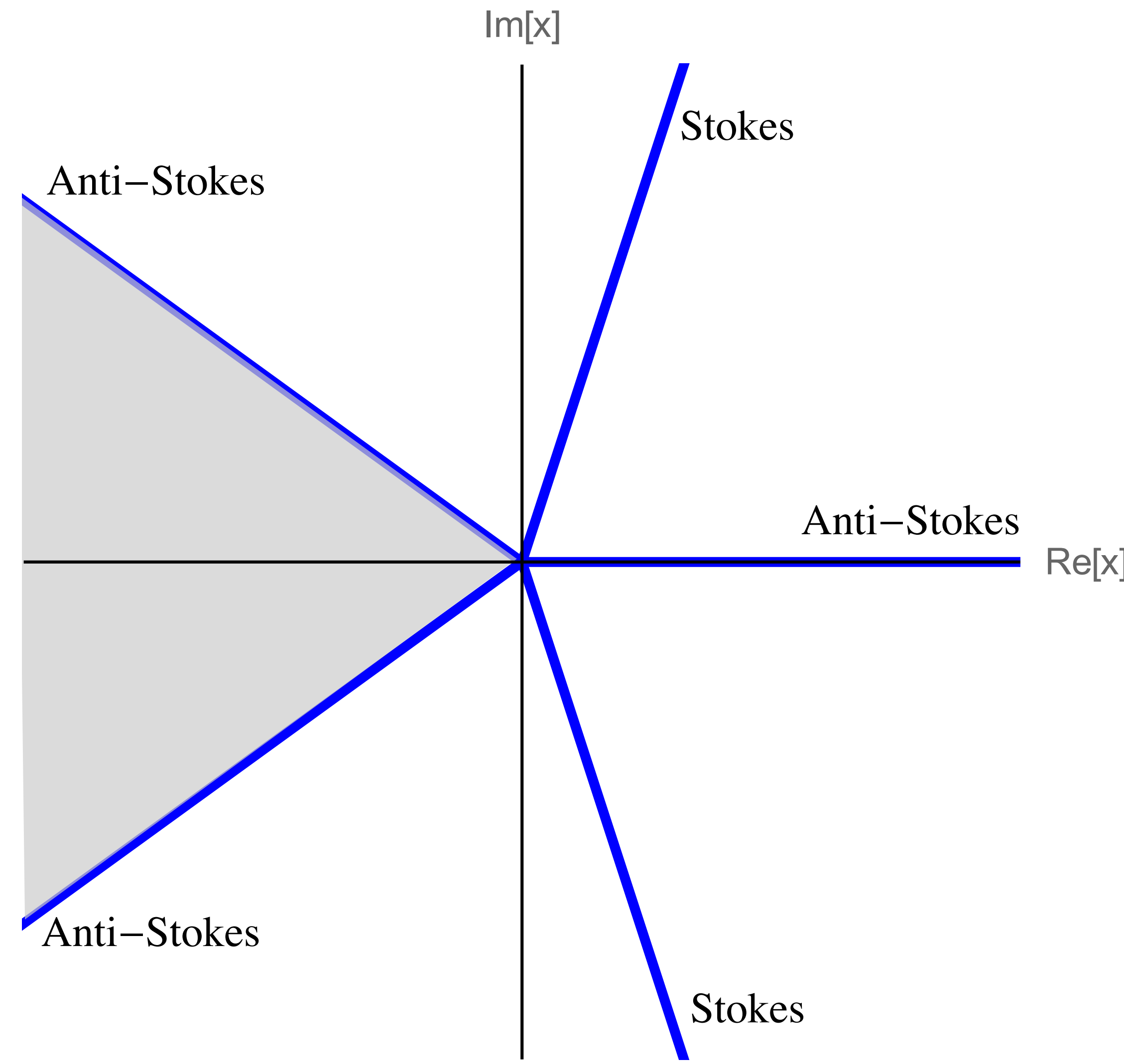
Zach Harris poster: Wed/Thurs

- how much “perturbative” information is required to detect, and to probe the properties of, a phase transition ?



Resurgent Extrapolation

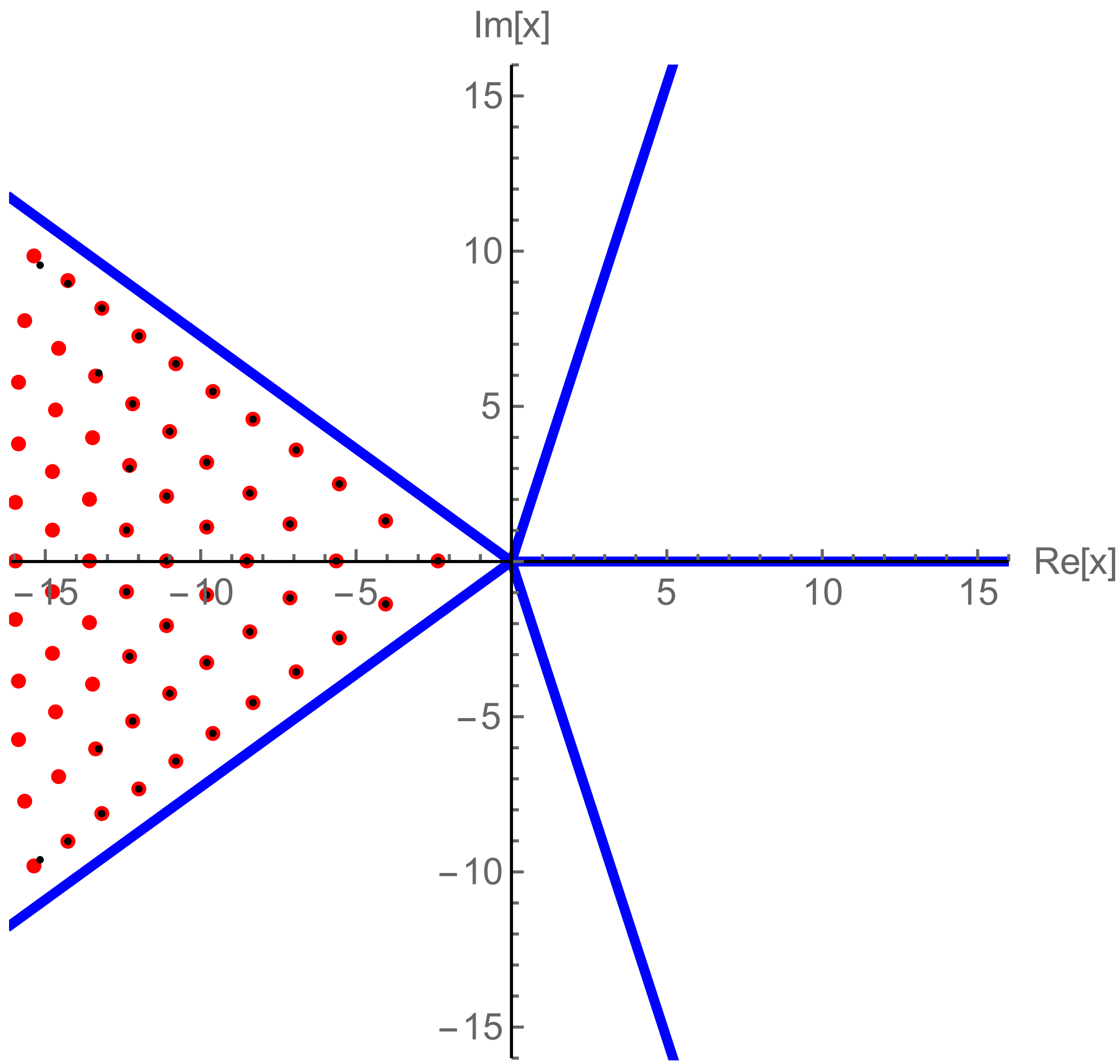
- case-study: Painleve I equation $\qquad y''(x) = 6\,y^2(x) - x$



- Painleve I equation has 5 sectors in the complex x plane, separated by phase transitions
- *tritronquée* solution: poles only in shaded region
- suppose we expand about x=+infty to finite order N: how much do these coefficients "know" about the other sectors?

Resurgent Extrapolation

- Pade-Borel + conformal or uniformizing maps: extreme precision
- 10 terms at x=+infty encode 23 digits of precision at x=0



Costin, GD: 1904.11593, 2009.01962

- extrapolate across Stokes transitions, even into the *tritronquée* pole region
- resurgent extrapolation can decode global behavior from surprisingly little input data from some other regime

Resurgent Extrapolation

- resurgent extrapolation can decode global behavior
- transmutation of the trans-series:
- near $x \to +\infty$ $\quad y(x) \sim -\sqrt{\frac{x}{6}} \sum_{n=0}^{\infty} a_n \frac{1}{x^{5n/2}} \quad , \quad x \to +\infty$
- along Stokes/anti-Stokes lines: exponentials
- into the pole region: $\frac{4\pi}{5} \leq \arg(x) \leq \frac{6\pi}{5}$

$$y(x) \approx \frac{1}{(x - x_{\text{pole}})^2} + \frac{x_{\text{pole}}}{10}(x - x_{\text{pole}})^2 + \frac{1}{6}(x - x_{\text{pole}})^3$$
$$+ h_{\text{pole}}(x - x_{\text{pole}})^4 + \frac{x_{\text{pole}}^2}{300}(x - x_{\text{pole}})^6 + \dots$$

- this phase transition is encoded in (few) fluctuation coeffs at $x = +\infty$
- resurgence: new summation & extrapolation methods

Conclusions

- "resurgence" is a new and improved form of asymptotics
- deep connections between perturbative and non-perturbative physics
- recent applications to differential eqs, QM, QFT, string theory, …
- 2-parameter trans-series can describe phase transitions
- outlook: new theoretical approach to quantum systems in extreme conditions
- outlook: computational definition of real-time path integrals
- outlook: computational access to strongly-coupled systems, phase transitions, particle production, and far-from-equilibrium physics