Bions in Large-N Sigma Models

Toshiaki Fujimori (Keio University)

based on arXiv:1607.04205, arXiv:1702.00589 arXiv:1705.10483, arXiv:1810.03768 …

In collaboration with

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Muneto Nitta **Norisuke Sakai**

Potential Toolkit to Attack Nonperturbative Aspects of QFT

September 24, 2020

Path Integral and Trans-series

path integral

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non-perturbative saddle points

$$Z_\sigma(g) = e^{-S_\sigma(g)} g^{\alpha_\sigma} \left(c_{\sigma,0} + c_{\sigma,1}\, g + c_{\sigma,2}\, g^2 + \cdots\right)$$

$$Z(g) = \sum_{\sigma \in \mathfrak{S}} n_\sigma Z_\sigma$$ **··· contribution of saddle point σ**

Saddle Points of Action

- **saddle points : solutions of equation of motion of S**

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→ "perturbative vacuum", "instanton", etc

Lefschetz Thimble

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Which saddle points?

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Renormalon

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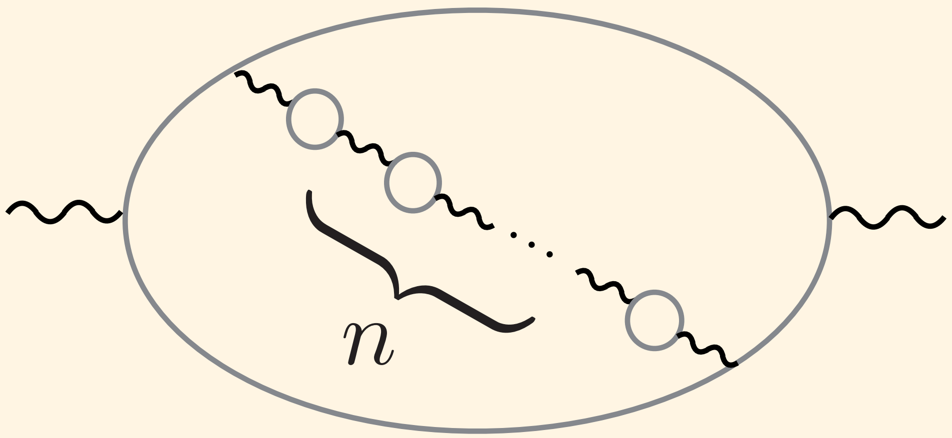
- **4d QCD, 2d NLSM, etc : diagrams with factorially large contribution**



$\sim n!$

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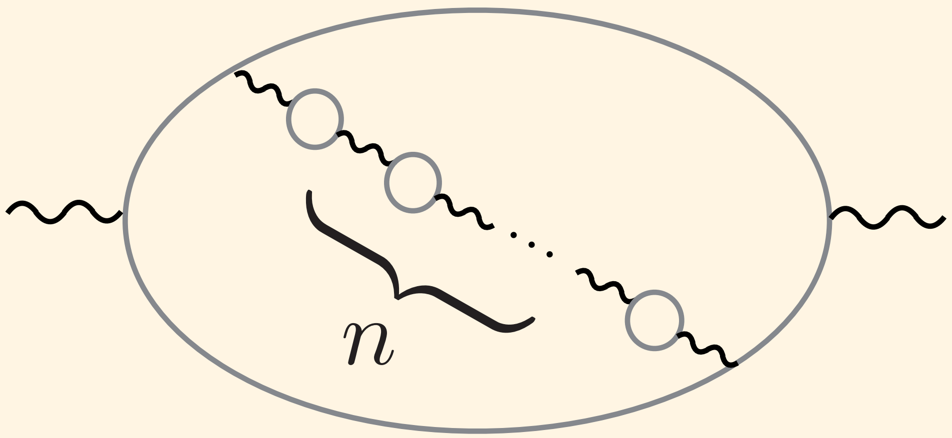

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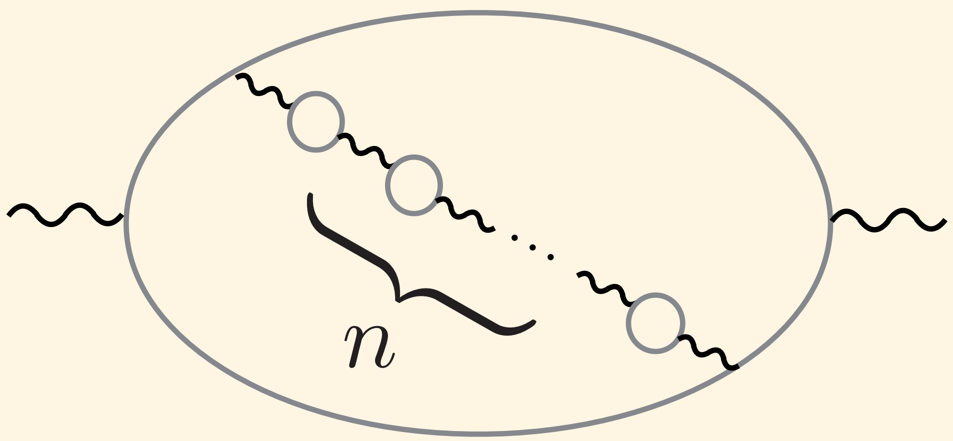

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→ **other saddle points?** **renormalon**

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relation between bion and renormalon?

[Argyres-Ünsal, 2012]

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↓

- 2d CP^N model is a good testing ground

2d analog of 4d gauge theory

2d CP^N model : toy model 4d QCD-like models

common properties : asymptotic freedom, instanton, large-N limit …

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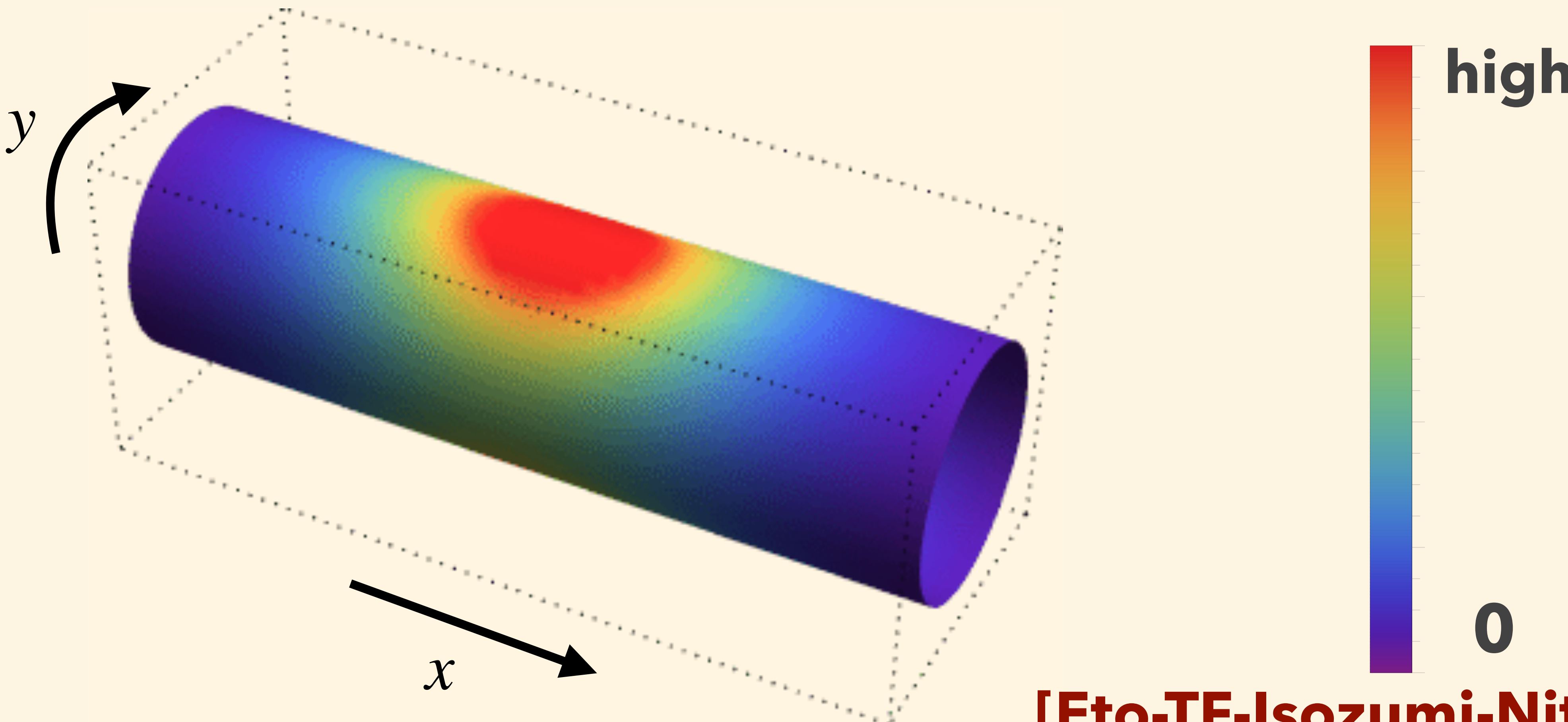
bion : non-trivial resurgence structure?

Fractional Instanton

- Euclidean action density of instanton in CP¹ model on $\mathbb{R} \times S^1$



[Eto-TF-Isozumi-Nitta-Ohashi-Ohta-Sakai, 2006]

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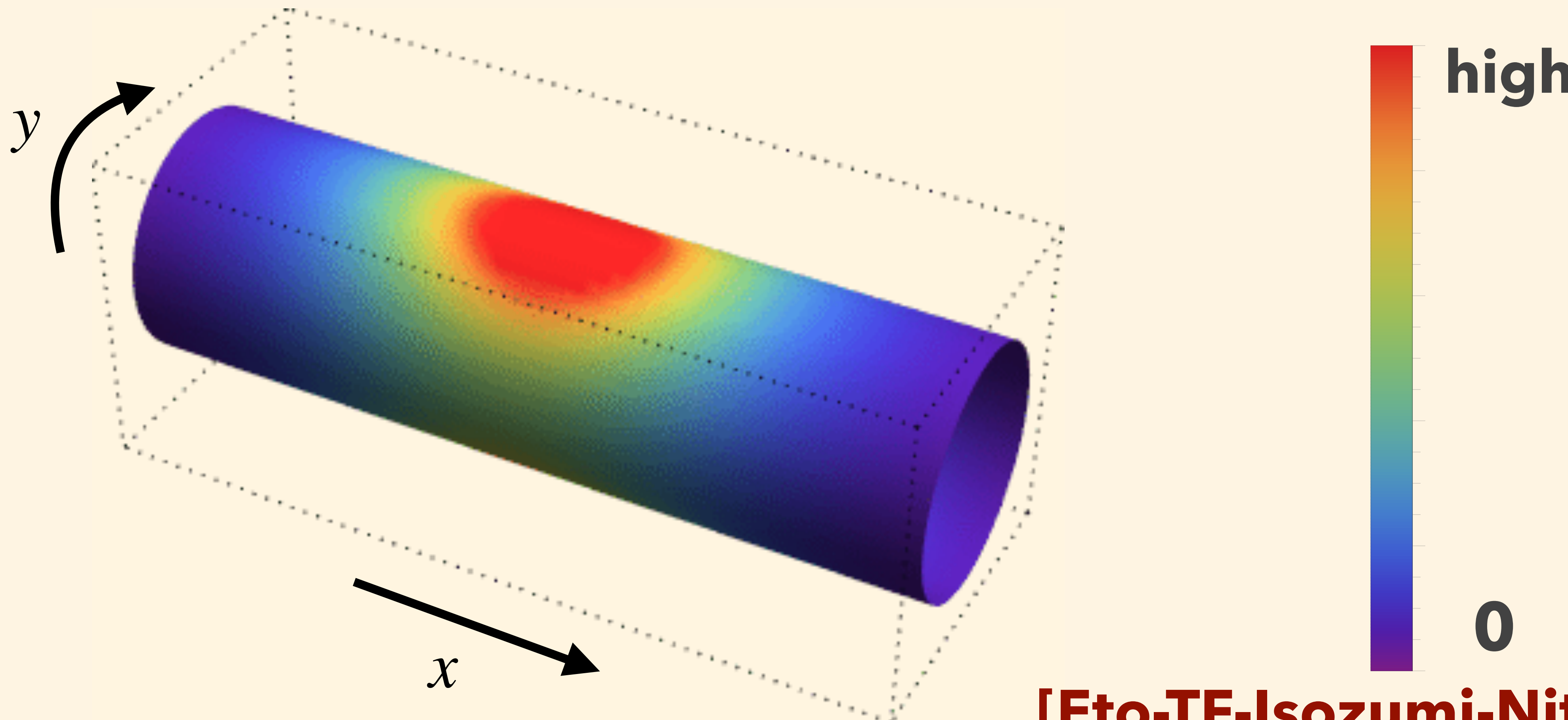

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CP^{N-1} Model on Cylinder

- cylinder : Euclidean time $\tau \in \mathbb{R}$ and spacial S^1

$x \sim x + 2\pi R$ $\quad$ R : compactification radius

$$\boxed{S = \sum_{A=1}^{N} \int d\tau dx \left[|\mathscr{D}_i \phi_A|^2 + iD\left(|\phi_A|^2 - \frac{1}{g^2 N} \right) \right]}$$

ϕ_A $(A = 1, \cdots, N)$: charged scalar fields $(q_A = 1)$

$\mathscr{D}_i$: covariant derivative $\mathscr{D}_i \phi_A = (\partial_i + iA_i)\phi_A$

A_i : auxiliary $U(1)$ gauge field, $\quad$ D : Lagrange multiplier

Twisted Boundary Condition

$$\phi_A(x + 2\pi R) \ = \ \exp(2\pi i R m_A)\, \phi_A(x)$$

Background Holonomy for SU(N) Global Symmetry

$$(m_1, \cdots, m_N) \ \in \ \mathfrak{u}(1)^{N-1} \ \subset \ \mathfrak{su}(N)$$

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$$m_A = \frac{A}{N}\frac{1}{R}$$ **: enhancement of Z_N-symmetry**

$$\phi_A \to e^{\frac{ix}{NR}}\,\phi_{A+1} \qquad A_x \to A_x - \frac{1}{NR}$$

➡ **adiabatic continuity**

Instantons in CP^{N-1} Model

- **single instanton (in the A-th classical vacuum)** $\quad u = e^{\frac{\tau + ix}{R}}$

$$\phi^B = \frac{1}{g} e^{-\frac{1}{2}\psi} u^{-m_B R} \times \begin{cases} u - u_0 & (B = A) \\ a_B & (B \neq A) \end{cases}$$

$$A_\tau = \frac{1}{2}\partial_x \psi, \quad A_x = -\frac{1}{2}\partial_\tau \psi, \quad D = \frac{i}{2}\partial_i^2 \psi$$

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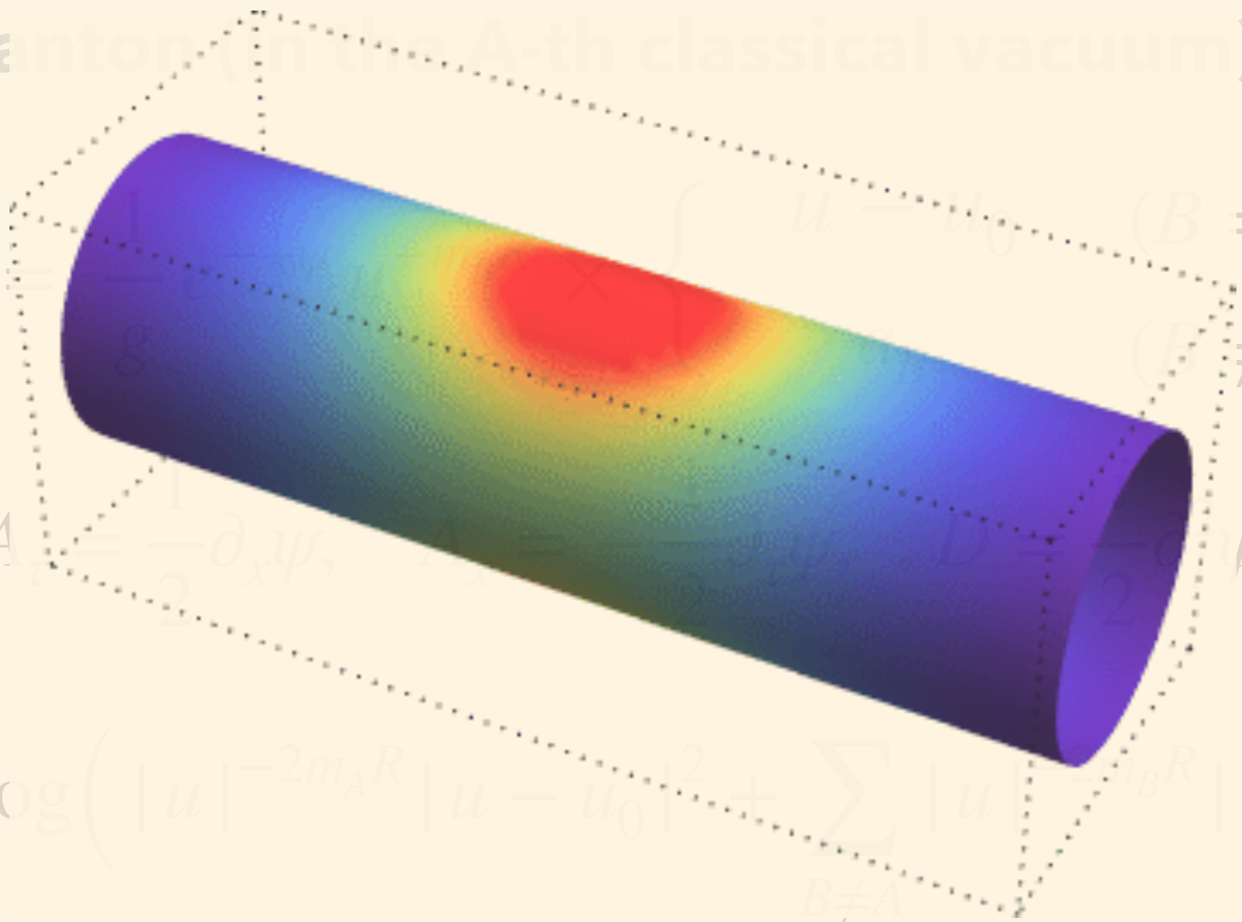

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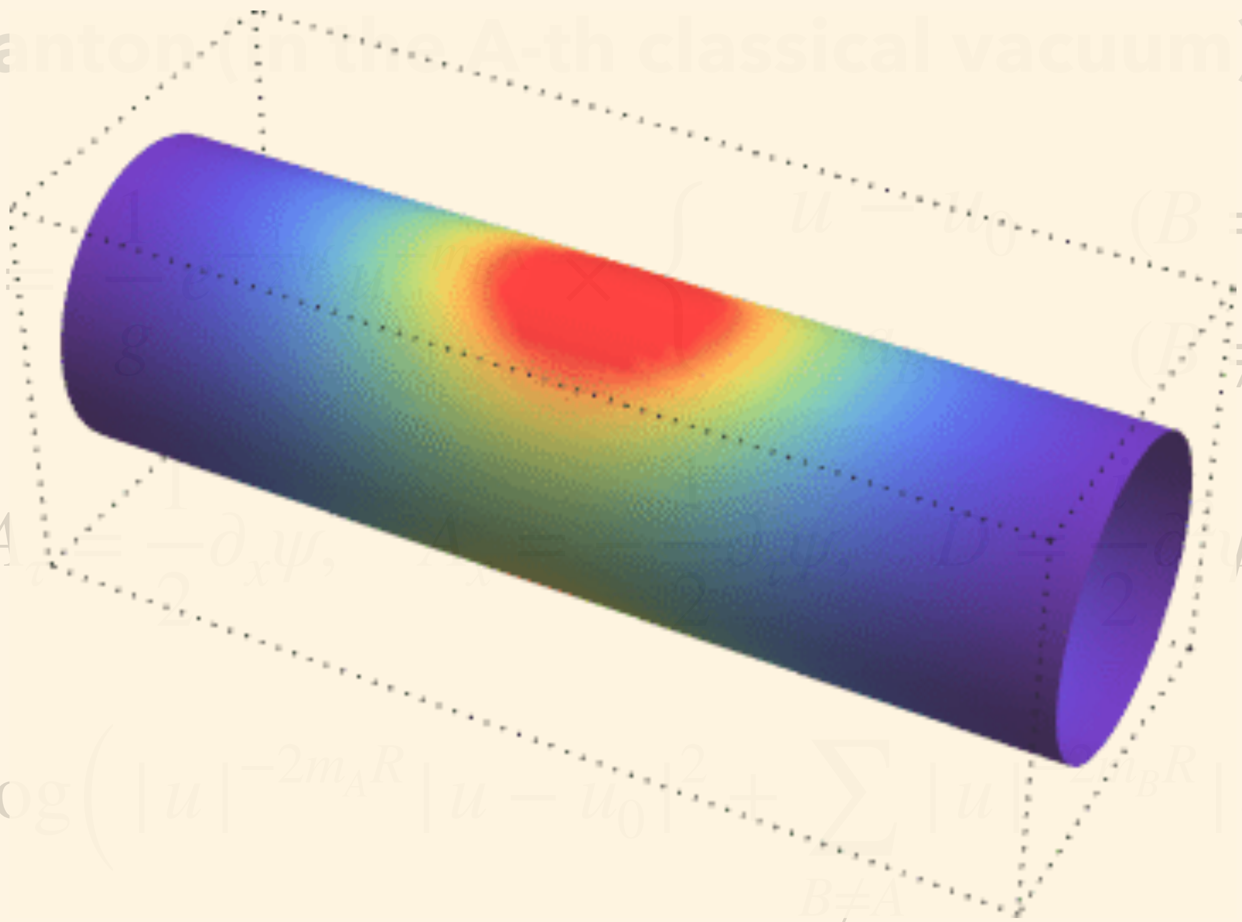

- **value of action for single instanton**

$$S = \frac{2\pi}{g^2} \quad \neq \text{ IR renormalon}$$

Fractional Instantons

- **fractional instanton solutions (N solutions A=1,⋯,N)**

$$\phi_A = e^{-\frac{1}{2}\psi} u^{-m_A R} \times a \qquad \phi_{A+1} = e^{-\frac{1}{2}\psi} u^{-m_{A+1} R}$$

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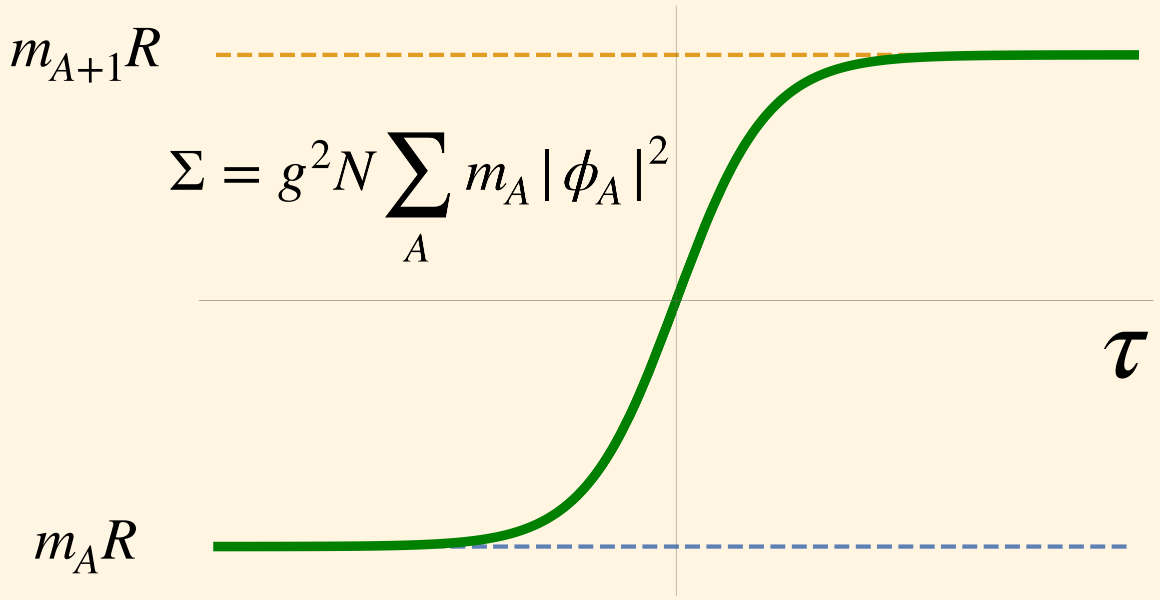

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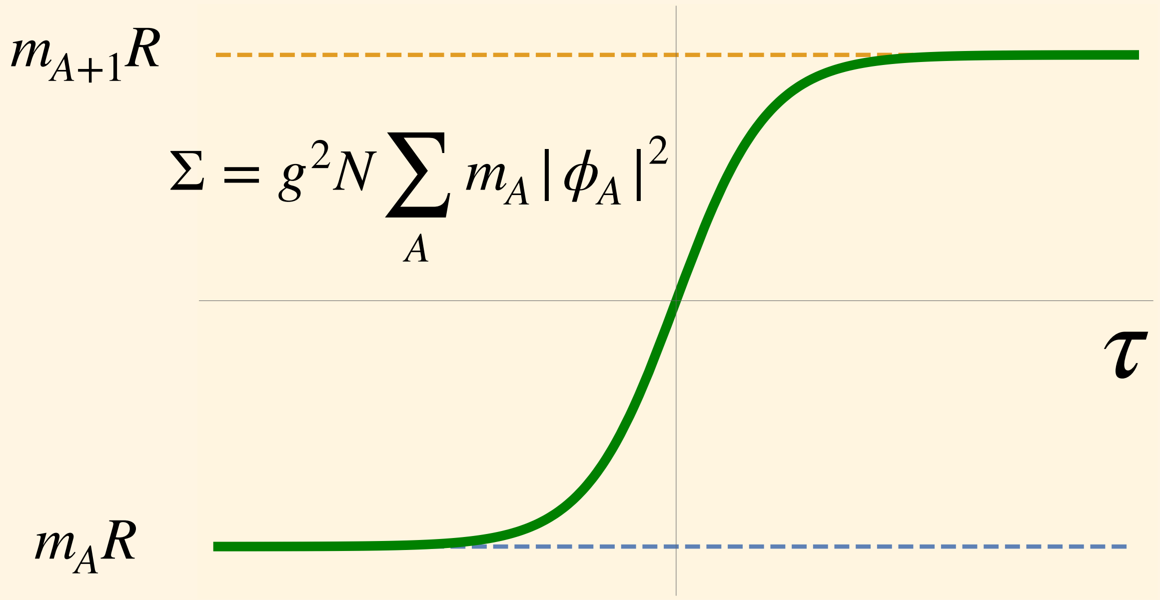

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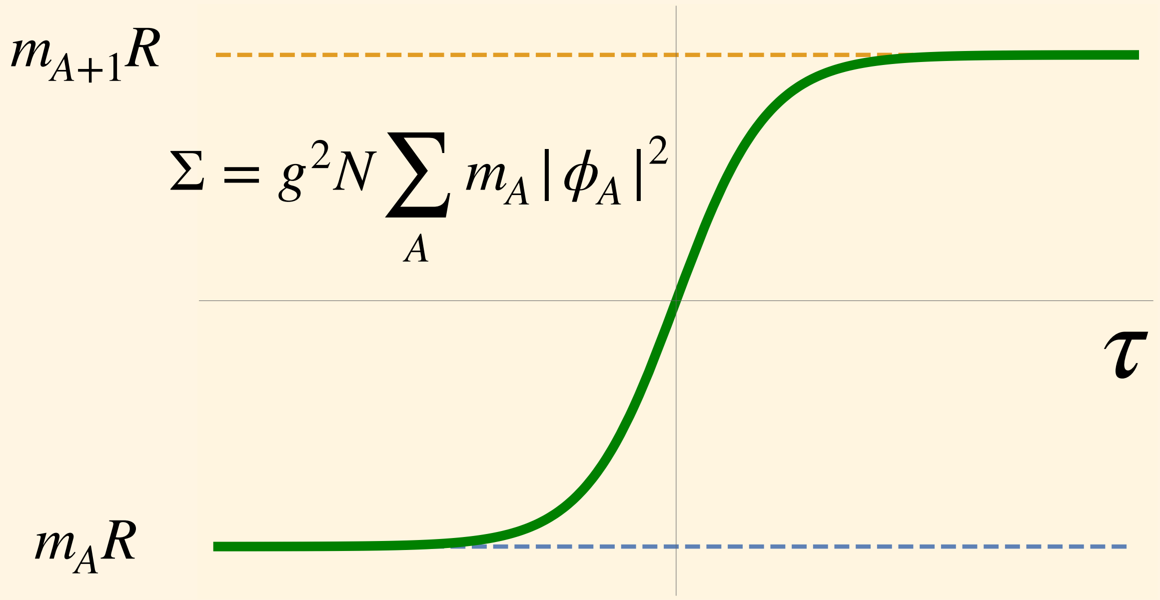

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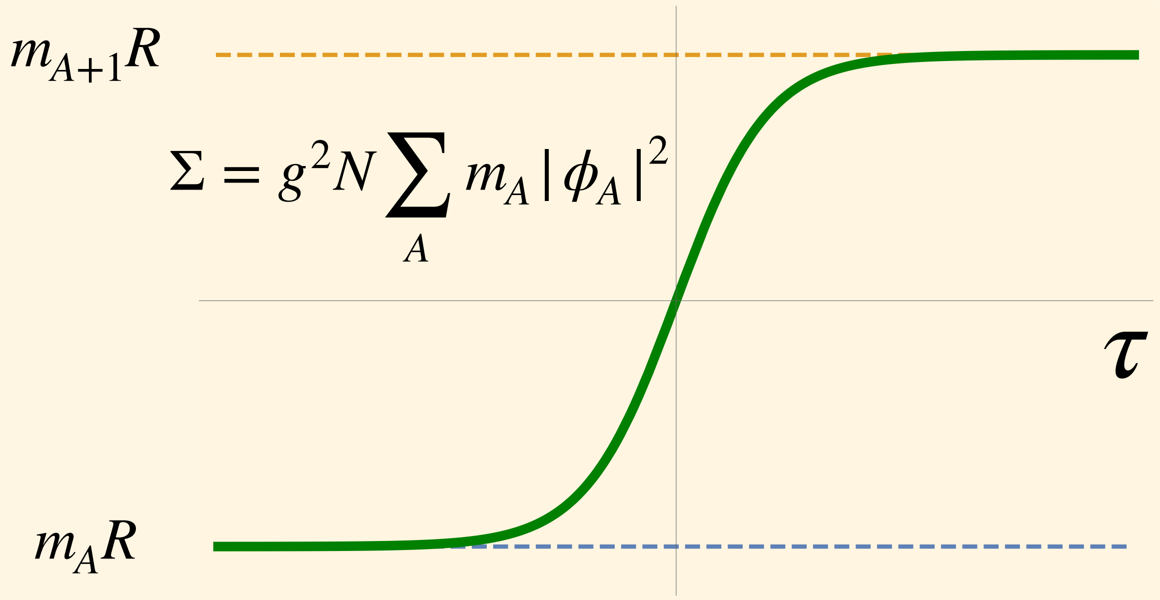

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in Z_N symmetric case $\quad S = \dfrac{2\pi}{g^2 N} \quad$ **(1/N × single instanton)**

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- **fractional instantons ··· no contribution to partition function**

$$Z = \mathrm{tr}\, e^{-\beta H}$$ **: periodic boundary condition** $\tau \sim \tau + \beta$

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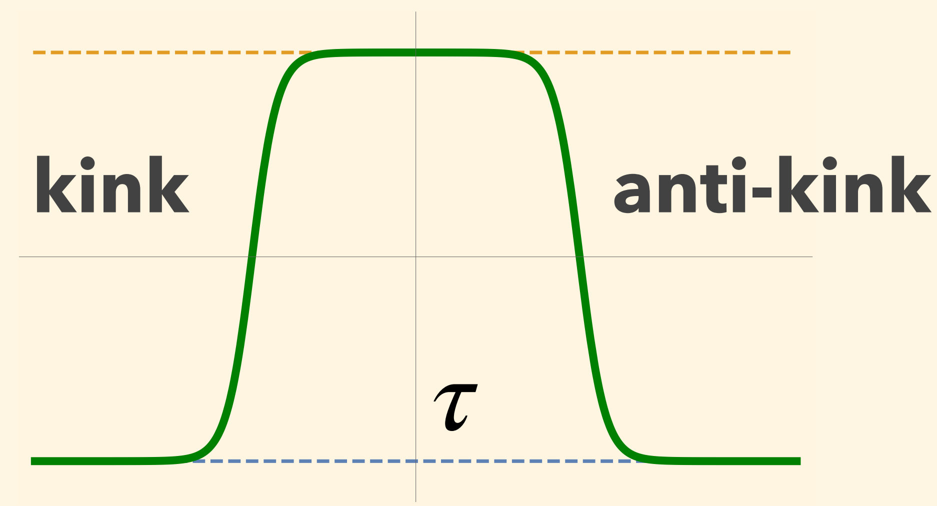

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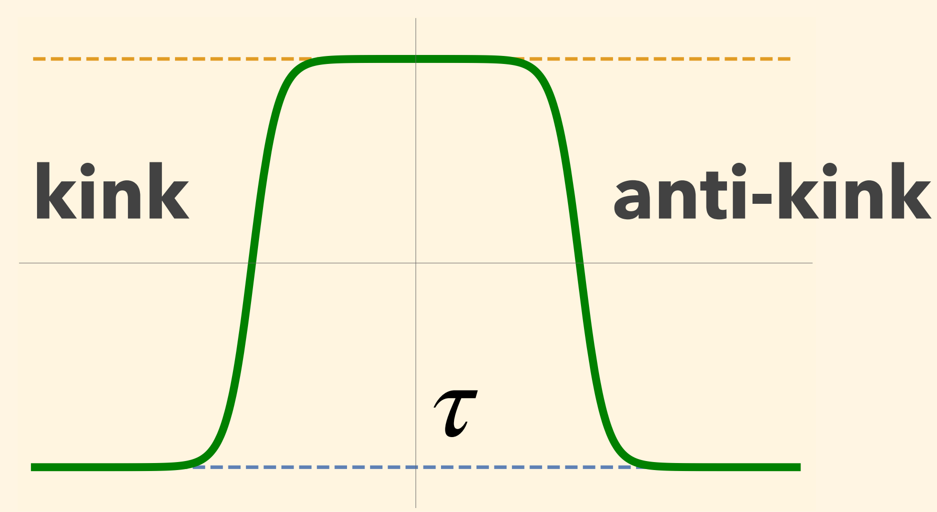

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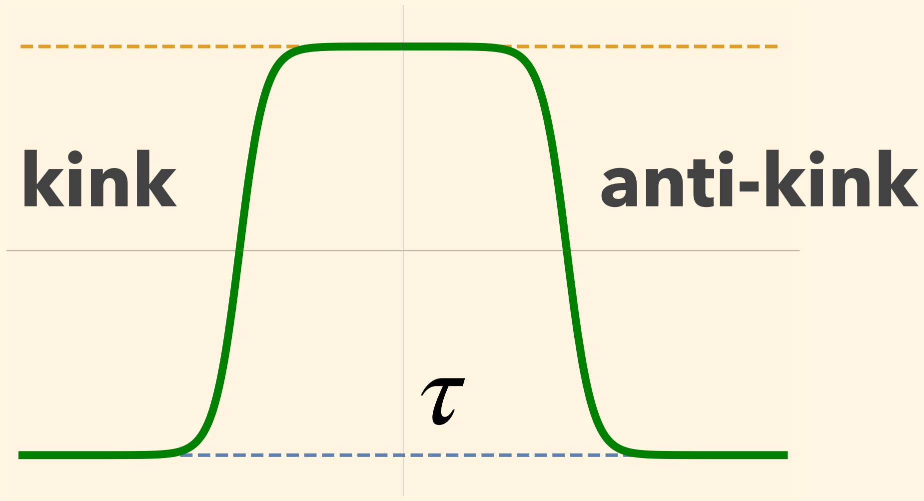

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Question : relation between bion and renormalon ?

Large-N Analysis

large-N bion contributions : two routes

1. g expansion ··· trans-series large-N limit
2. large-N limit 't Hooft coupling expansion

[Ishikawa, Morikawa, Nakayama, Shibata, Suzuki, Takaura, 2019]

ambiguity of $\langle F^2 \rangle \sim \Lambda^3$ on R×S¹ with TBC (route 2)

≠ n-bion contribution Λ^{2n} (route 1)

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Can we correctly capture
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Plan of Talk

Intoduction

Route 1 : Semi-classical (small g) Expansion

Route 2 : Large-N Analysis

Setup

2d $\mathcal{N} = (2,2)$ SUSY CP^{N-1} sigma model

$$\mathscr{L} = \sum_{A=1}^{N} \left[|\mathscr{D}_i\phi_A|^2 + |\sigma\phi_A|^2 + iD\left(|\phi_A|^2 - \frac{1}{g^2 N} \right) + \text{ fermions } \right] + \mathscr{L}_{\text{source}}$$

- **model on torus** $(\tau, x) \sim (\tau + \beta, x) \sim (\tau, x + 2\pi R)$
- **periodic boundary condition for** $\tau \sim \tau + \beta \quad (\beta \to \infty)$

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$$\phi^A(x + 2\pi R) \;=\; \exp(2\pi i m^A R)\, \phi^A(x)$$

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n-point correlation function $E^{(n)} = \frac{1}{n!}\left(\frac{\partial}{\partial \epsilon}\right)^n \left[-\lim_{\beta\to\infty}\frac{1}{\beta}\log Z\right]_{\epsilon=0}$

$E^{(1)} = \langle \Sigma \rangle \qquad E^{(2)} = \int d\tau \, \langle \Sigma(0)\,\Sigma(\tau)\rangle \quad \cdots$

Route 1
Semi-classical Expansion
(Small Coupling Expansion)

Eliminating Auxiliary Fields

- **e.o.m. for auxiliary vector multiplet** $(A_i\,, \sigma\,,\, D, \lambda)$

$$A_i = \frac{i}{2}\frac{\bar{\phi}_A \partial_i \phi_A - \phi_A \partial_i \bar{\phi}_A}{|\phi_A|^2} + \cdots \qquad \sigma = 0 + \cdots \qquad \textbf{etc}$$

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$$\mathscr{L} = \frac{2}{g^2}\left[G_{a\bar{b}}\left(D_\mu\varphi^a D^\mu\bar{\varphi}^{\bar{b}} - \bar{\psi}_l^{\bar{b}}\,D_z\psi_l^a - \bar{\psi}_r^{\bar{b}}\,D_{\bar{z}}\psi_r^a\right) + \frac{1}{2}R_{a\bar{b}c\bar{d}}\,\psi_l^a\bar{\psi}_l^{\bar{b}}\psi_r^c\bar{\psi}_r^{\bar{d}}\right] + \mathscr{L}_{\text{sorce}}$$

2d CP¹ Sigma Model

2d $\mathcal{N} = (2,2)$ SUSY CP¹ sigma model

$$\mathscr{L} = \frac{1}{g^2}\frac{1}{\left(1+|\varphi_a|^2\right)^2}\left[\,|\,\partial_i\phi\,|^2 - \bar{\psi}_r\mathscr{D}\psi_r - \bar{\psi}_l\bar{\mathscr{D}}\psi_l + \frac{\bar{\psi}_r\psi_r\bar{\psi}_l\psi_l}{\left(1+|\varphi_a|^2\right)^2}\right]$$

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$$(\tau, x) \sim (\tau, x + 2\pi R) \in \mathbb{R} \times S^1$$

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$$\varphi(x + 2\pi R) = e^{2\pi i m R}\,\varphi(x)$$

$$\psi_{l,r}(x + 2\pi R) = e^{2\pi i m R}\,\psi_{l,r}(x)$$

$$\mathscr{L}_{\text{sorce}} = \epsilon\,\frac{1-|\varphi|^2}{1+|\varphi|^2}$$

deformation term in Lagrangian

↓

$Z(\epsilon)$ **: generating function for "height"**

Equation of Motion

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- x-independent ansatz

$$\varphi(\tau, x) \to \varphi(\tau)$$

$$\tilde{\varphi}(\tau, x) \to \tilde{\varphi}(\tau)$$

Leading order contribution

Bion Solution

- two conserved charges (time shift, phase rotation)

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solution (up to sym.)

$$\varphi = \tilde{\varphi} = A\frac{\mathrm{sn}(\omega\tau, k)}{\mathrm{cn}(\omega\tau, k)}$$

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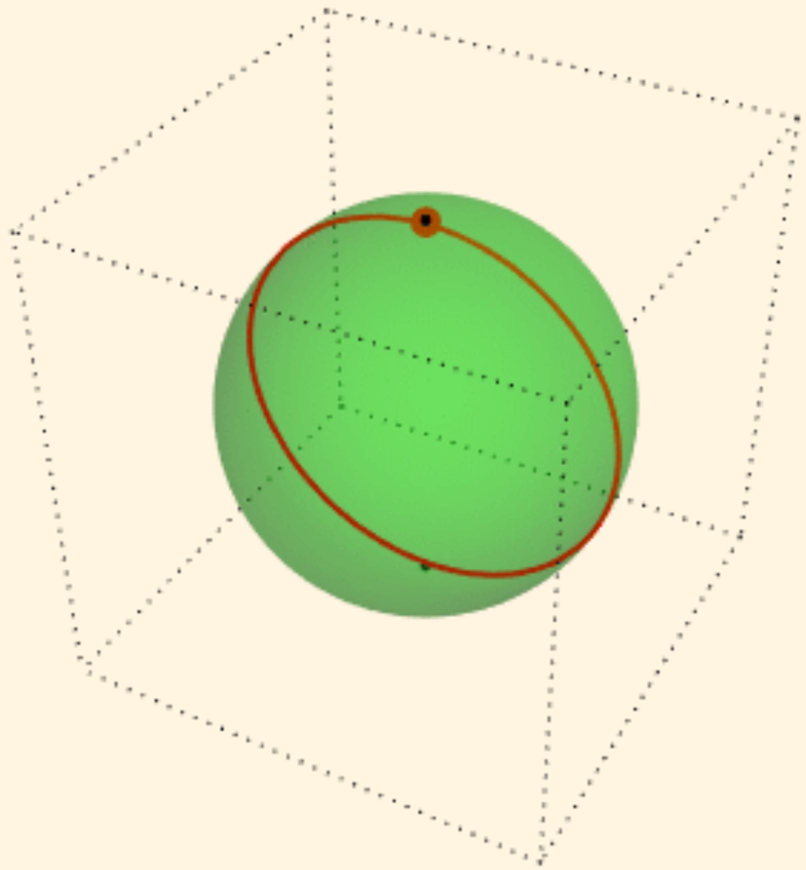
- complex saddle points $\tilde{\varphi} \neq \bar{\varphi}$ for generic p, q

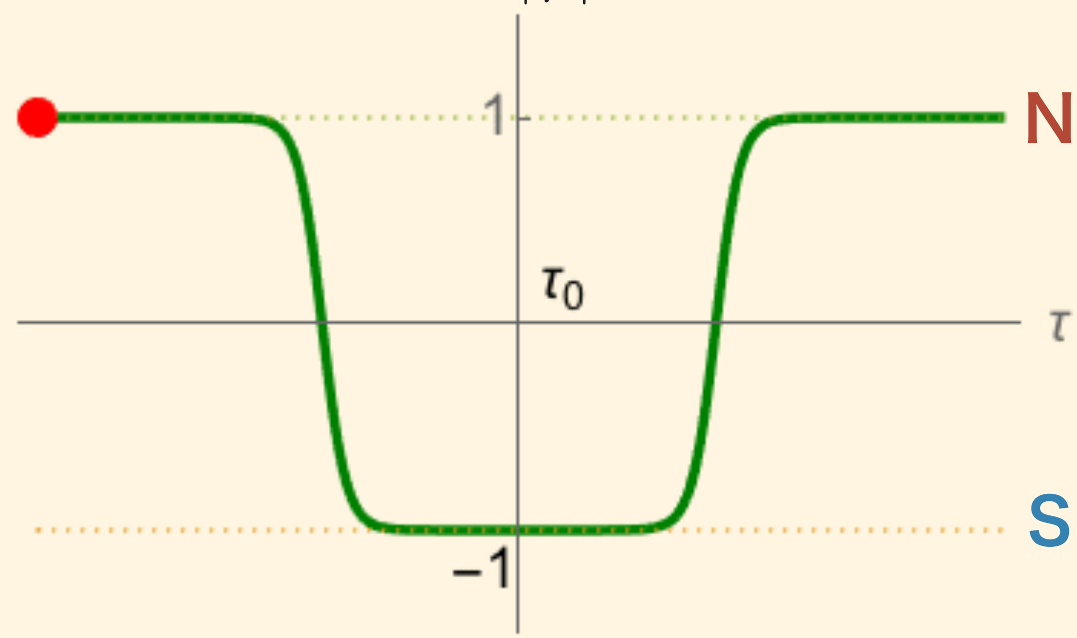
Real Bion Solution (p=1)

- **single bion solution**

$$\varphi = \tilde{\varphi} = A \sinh \omega\tau \quad (\beta \to \infty)$$



$$\Sigma = \frac{1 - |\varphi|^2}{1 + |\varphi|^2}$$ **: height**



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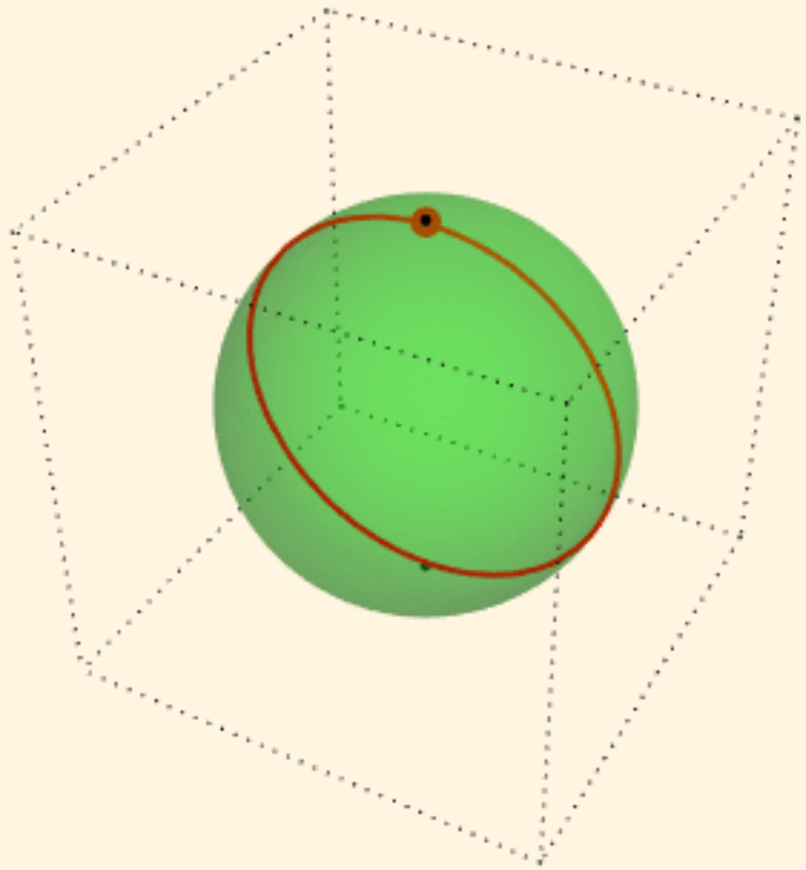

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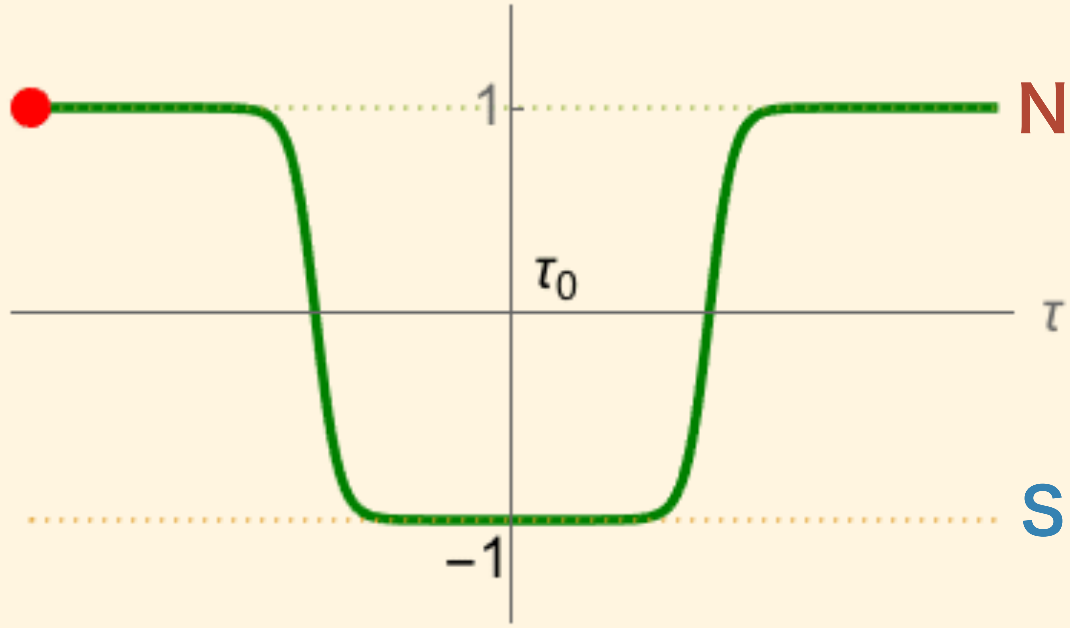

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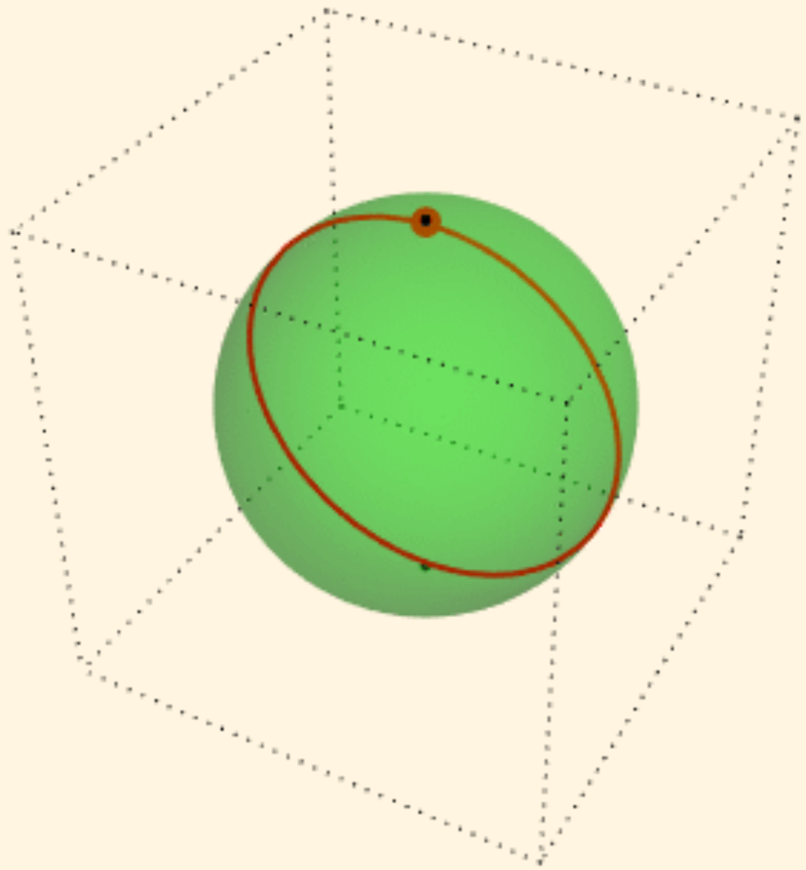

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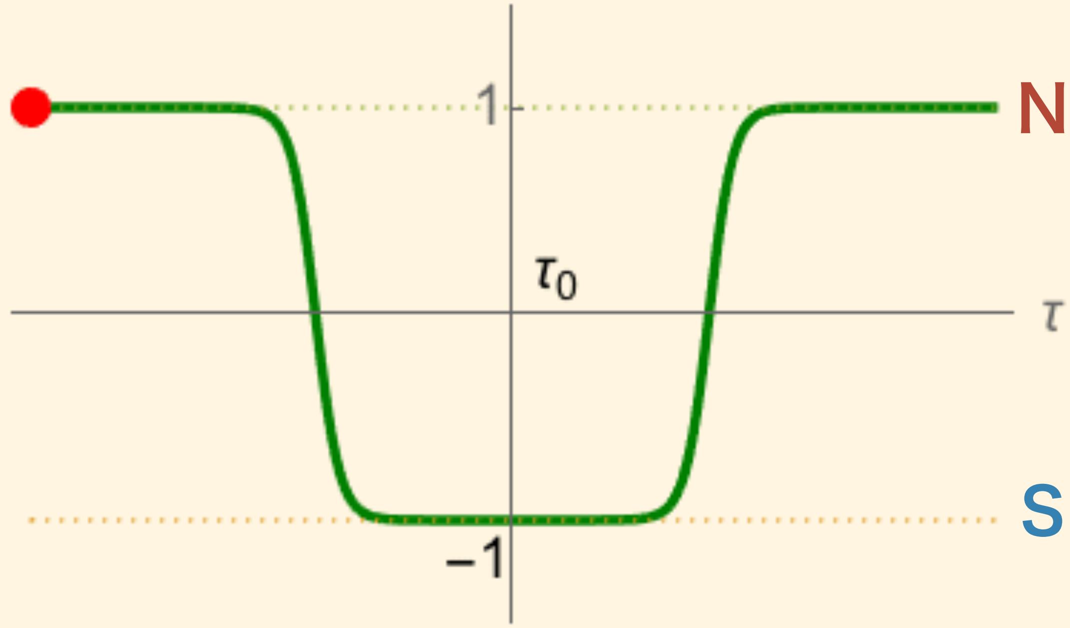

$$E^{(0)}_{\mathrm{real}} \sim e^{-\frac{2m}{g^2}} \neq 0$$

even for $\epsilon = 0$

there must be another contribution

Complex Bion Solution (p=1)

complex bion :
not on the original contour

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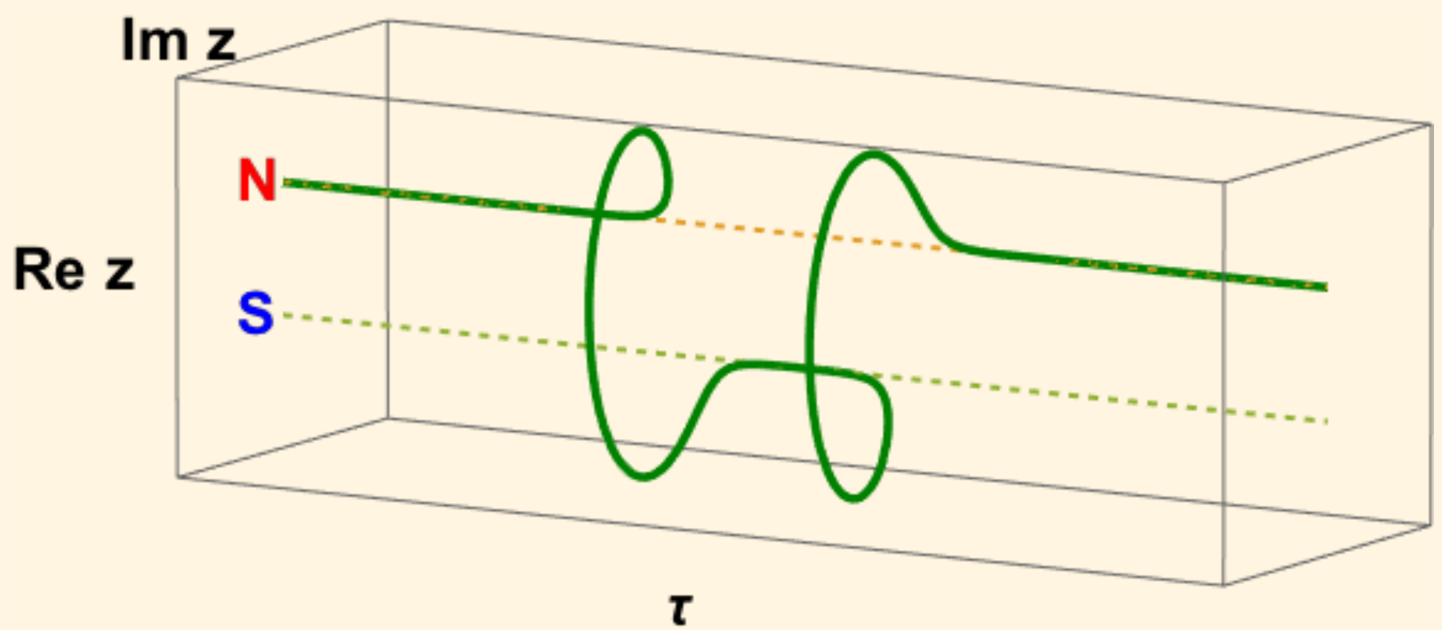

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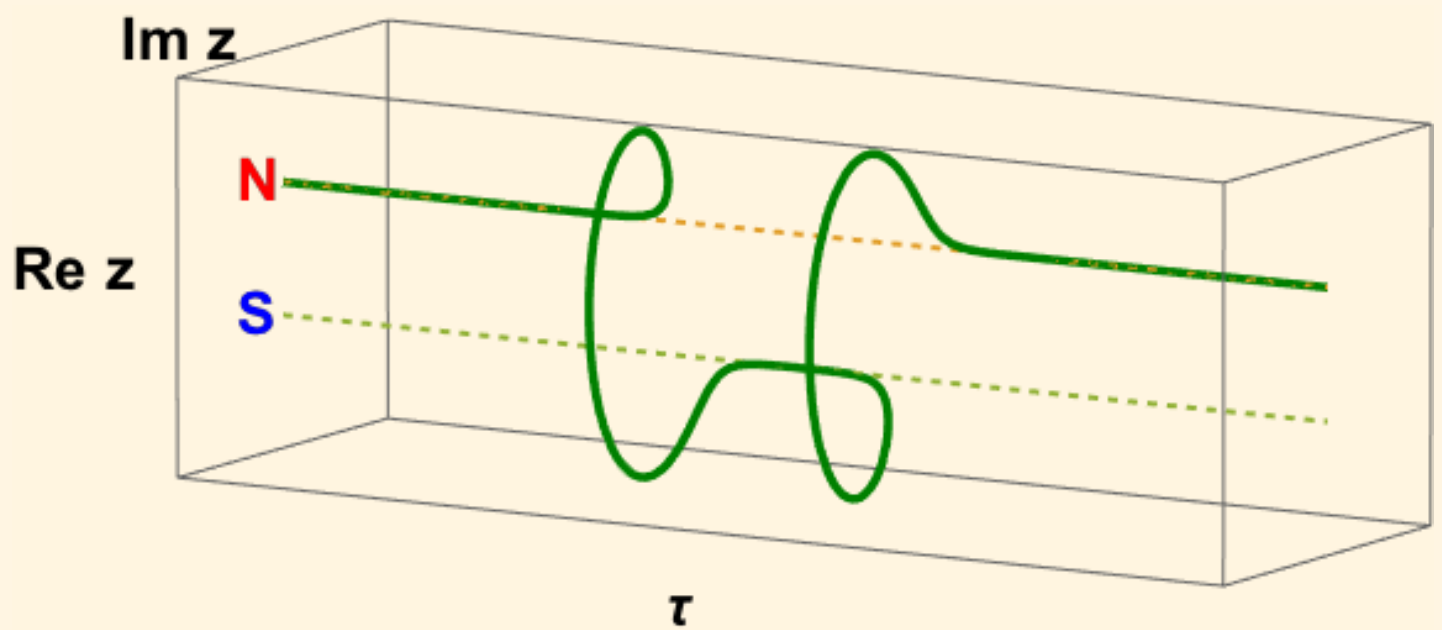

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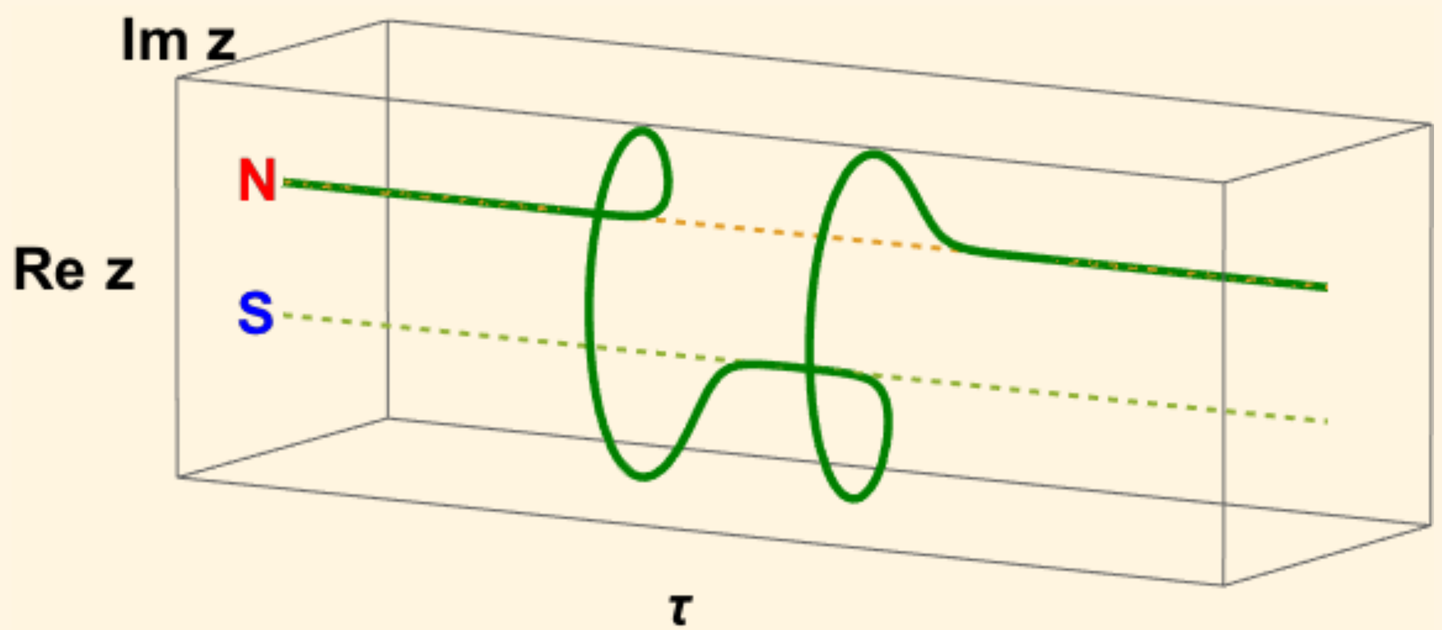


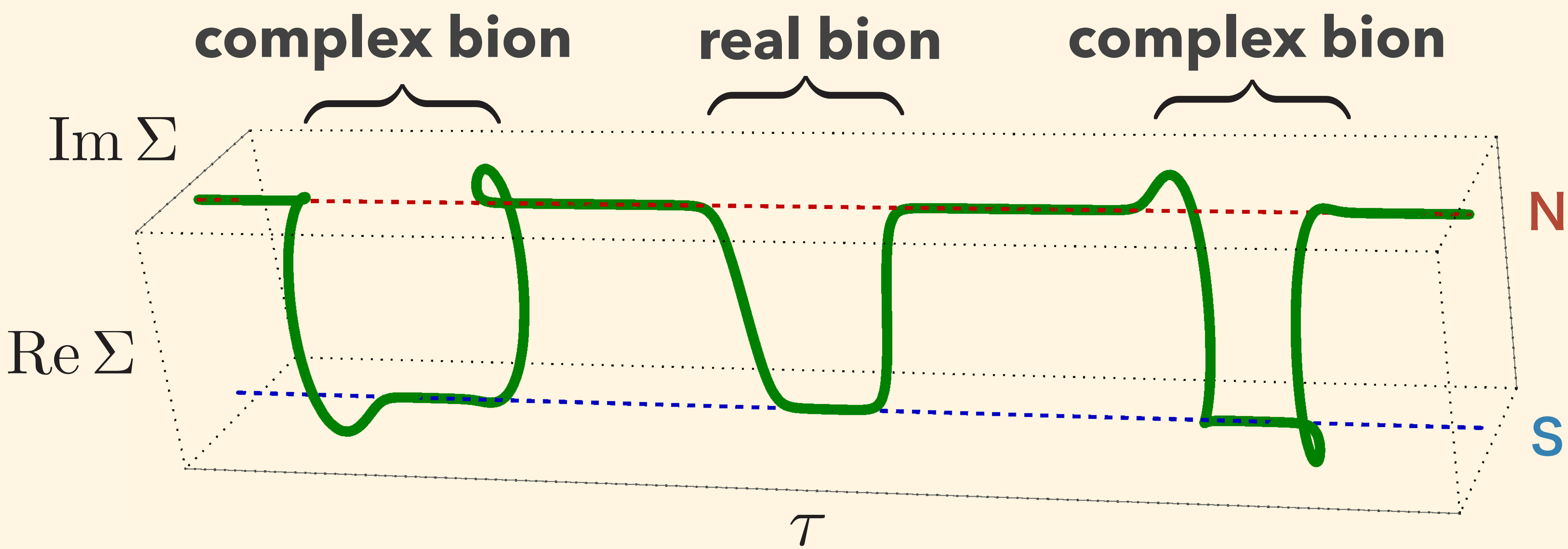
$$E^{(0)}_{\rm real} + E^{(0)}_{\rm complex} = 0$$

for $\epsilon = 1$

consistent with SUSY

Kink profile of multi-bion

- p : number of bions
- q : label saddle points in p-bion sector



$(p, q) = (3, 1)$: multi bion solution

Quasi-Moduli Integral

$$Z_p = \int d\mathrm{vol}\, \frac{\det \Delta_F}{\det \Delta_B} \exp(-S_{\mathrm{eff}}) + \mathcal{O}(g^2)$$

one-loop determinant

bion effective action

"nearly flat" directions

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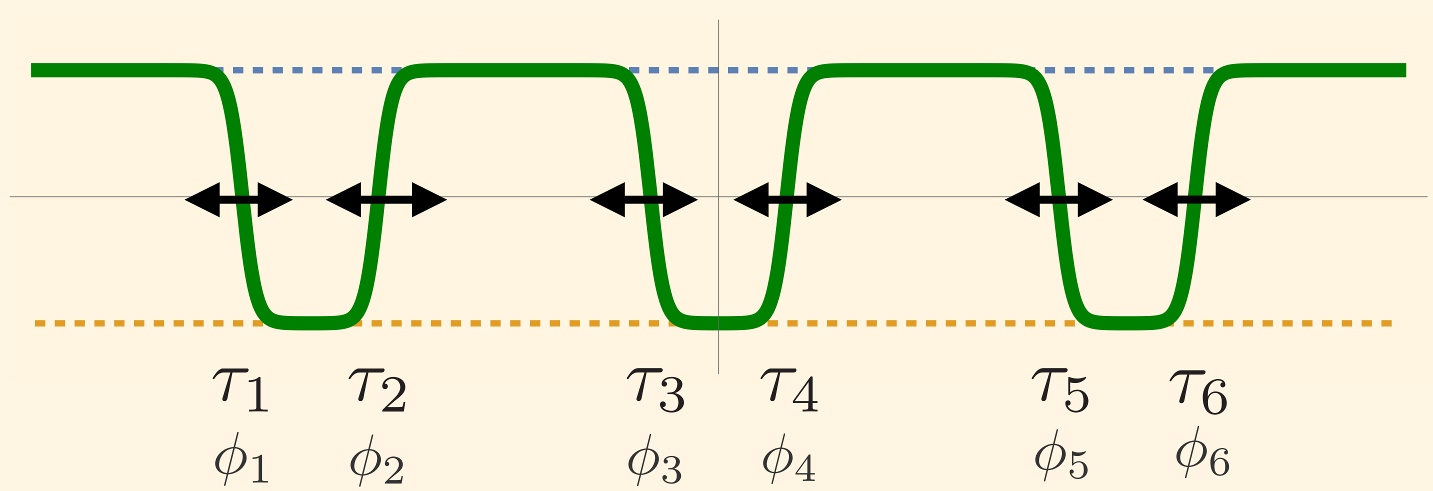

kink gas with complexified quasi-moduli

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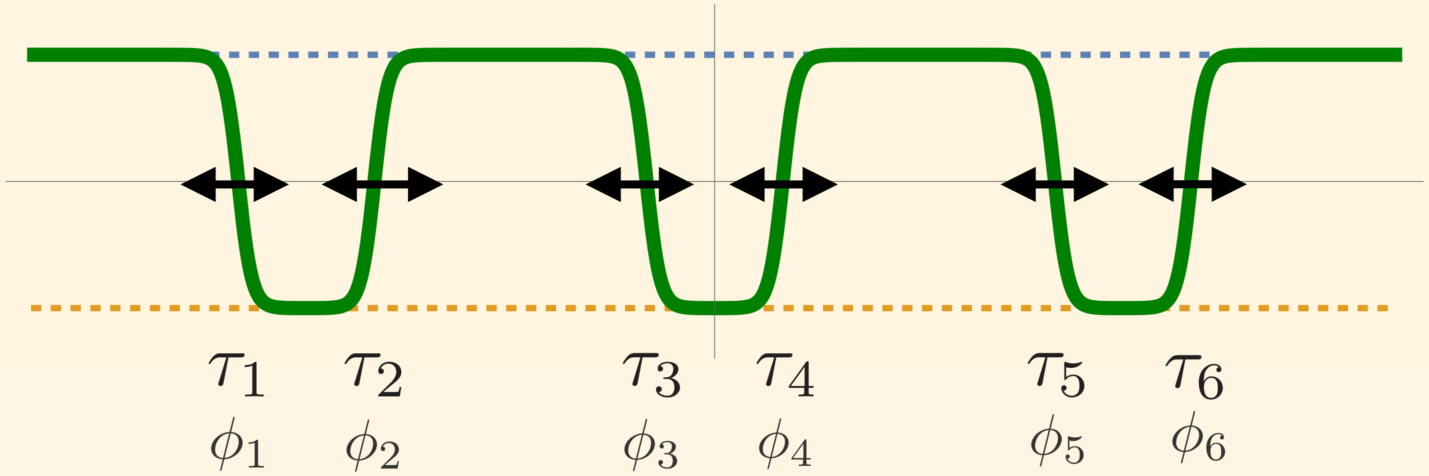

kink gas with complexified quasi-moduli

Single Bion Contribution

- **quasi-moduli integral for p-bion contribution**

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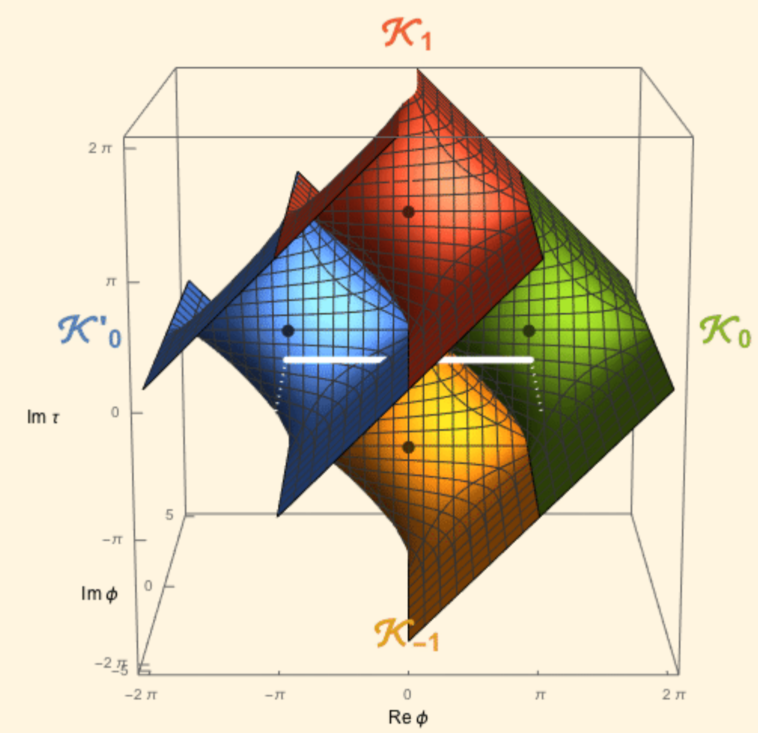

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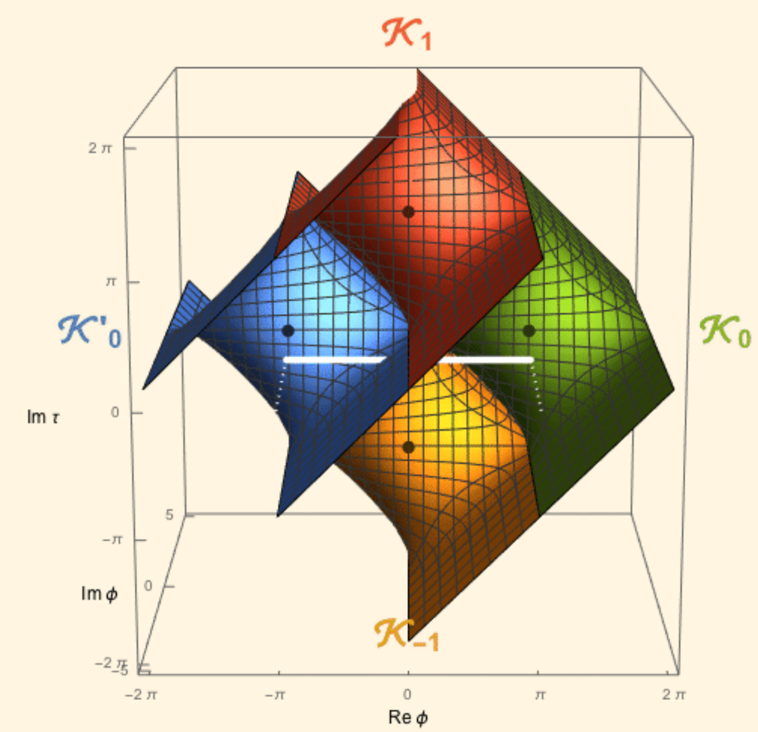

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- **single bion contribution to** $E = -\lim_{\beta\to\infty} \frac{1}{\beta} \log Z$

$$\boxed{E_{p=1} = -2m \frac{\Gamma(\tilde{\varepsilon})}{\Gamma(1-\tilde{\varepsilon})} \left[\frac{4\pi mR}{g_R^2} \frac{\Gamma(1-mR)}{\Gamma(1+mR)} \right]^2 \left(\frac{4\pi mR}{g_R^2} \right)^{-2\tilde{\varepsilon}} e^{\mp \tilde{\varepsilon}\pi i}\, e^{-\frac{4\pi mR}{g^2}} + \cdots}$$

with $\tilde{\varepsilon} = \varepsilon + 1$ **and** $\frac{1}{g_R^2} = \frac{1}{\pi} \log |R\Lambda|$

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imaginary ambiguity

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Generalization to CP^{N-1} model

N-1 types of bions

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↓ single bion contributions

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N-1 types of bions

single bion contributions

jump of imaginary part

$$\Delta E_{p=1} \;=\; i\sum_{b=1}^{N-1}\left[2\pi m_b \mathscr{A}_b \left(\frac{1}{\Gamma(1-\tilde{\varepsilon})}\frac{\Gamma(1+m_bR)}{\Gamma(1-m_bR)}\right)^2 \left(\frac{4\pi m_b R}{g_R^2}\right)^{2(1-\tilde{\varepsilon})}\right]\exp\left(-\frac{4\pi m_b R}{g_R^2}\right)$$

$$\mathscr{A}_b = \prod_{a\neq b}\frac{m_a}{m_a-m_b}\frac{\Gamma(1+(m_a-m_b)R)}{\Gamma(1-(m_a-m_b)R)}\frac{\Gamma(1-m_bR)}{\Gamma(1+m_bR)}$$

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$$=\; 2NR\,|\,\Lambda\,|^2\left[\; -\,\tilde{\varepsilon} + \left(\log\frac{4\pi}{g_R^2 N} + \gamma \pm \underline{\frac{\pi i}{2}}\right)\tilde{\varepsilon}^2 + \cdots \;\right]$$

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fixed in the large-N limit

survives in the large-N limit

1d limit ⋯ CP^{N-1} QM

ground state energy

$$E^{(0)} = 0$$

one-point function

$$E^{(1)} = -m\coth\frac{m}{g^2} + g^2$$

two-point function

$$E^{(2)} = E^{(2)}_{\text{pert}} - \sum_{p=1}^{\infty} e^{-\frac{2m}{g^2}p}\left[4mp^2\left(\pm\frac{\pi i}{2} + \gamma + \log\frac{2m}{g^2}\right) + \mathcal{O}(g^2)\right]$$

complete agreement with exact results

obtained from Schrödinger equation

Route 2
Large-N Analysis

2d CP^{N-1} sigma model

2d $\mathcal{N} = (2,2)$ **SUSY CP^{N-1} sigma model**

$$\mathscr{L} = \sum_{A=1}^{N} \left[|\mathscr{D}_i \phi_A|^2 + |\sigma \phi_A|^2 + iD\left(|\phi_A|^2 - \frac{1}{g^2 N} \right) + \textbf{ fermions } \right] + \mathscr{L}_{\text{source}}$$

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effective action for vector multiplet

$$S_{\text{eff}} = -N \left[\frac{1}{N} \sum_{a=1}^{N} \log \operatorname{sdet} \left[\Delta(A_i, \sigma, D, \lambda, m_a, \epsilon) \right] - \frac{iD}{g^2N} \right]$$

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large-N limit : $N \to \infty$ **with** $g^2 N$ **… fixed**

Large-N Saddle Point

constant ansatz

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$$\left.\frac{\delta S_{\text{eff}}}{\delta D}\right|_{\text{const}} \propto \sum_{n=-\infty}^{\infty} \sum_{A=1}^{N} \frac{1}{R} \frac{1}{\sqrt{(n/R + m_A + A_x)^2 + |\sigma|^2 - g^2 N \epsilon m_A + iD}} - \frac{1}{g^2}$$

⋮

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$$\sum_{A=1}^{N} f(m_A) = \sum_{A=1}^{N} f\left(\frac{A}{NR}\right) \xrightarrow{\text{large-N}} NR \int_0^{\frac{1}{R}} dp\, f(p)$$

Generating Function

eom $\dfrac{\delta S_{\mathrm{eff}}}{\delta D} = 0 \quad \cdots$ ➡ **large-N saddle point solution**

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$E^{(0)} = S_{\text{sol}}(\epsilon = 0) = 0$ ··· **SUSY vacuum**

$E^{(1)} = \partial_\epsilon S_{\text{sol}}|_{\epsilon=0} = -2NR|\Lambda|^2$ ··· **agree with bion**

1d Limit ··· CP^{N-1} QM

$R \to 0$ **limit with** $m_A = \dfrac{A}{N} M$ **and** $\dfrac{1}{g_{1d}^2} = \dfrac{2\pi R}{g_{2d}^2}$ **··· fixed**

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Two Point Funciton in CP^{N-1} SQM

large-N analysis

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not consistent with bion and Schrdinger eq.

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bion and Schrödinger eq.

$$\frac{\partial^2}{\partial\epsilon^2}E_{\mathrm{pert}}(\epsilon=0) \;=\; N\left[g^2N - \Gamma(N+1)\int_0^\infty dt\, e^{-\frac{t}{g^2}}\frac{\Gamma(1-t\,N/2M)}{\Gamma(N-t\,N/2M)}\right]$$

Summary

- bion contributions in CP^{N-1} sigma model

- large-N limit with Z_N symmetric twisted boundary condition

- some of bion contributions are correctly reproduced

- non-trivial resurgence structure has not yet been reproduced

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→ **future problem**