Spheroidal Harmonics and Nekrasov Function

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w/ G. Aminov & A. Grassi

What?

- **In this talk, I focus on the spheroidal differential equation**

$$\left[\frac{d}{d\xi}(1-\xi^2)\frac{d}{d\xi}+(c\xi)^2-\frac{m^2}{1-\xi^2}+\lambda\right]S(\xi)=0$$

or its generalization

$$\left[\frac{d}{d\xi}(1-\xi^2)\frac{d}{d\xi}+(c\xi)^2-2cs\xi+s-\frac{(m+s\xi)^2}{1-\xi^2}+\lambda\right]S(\xi)=0$$

Where?

- Such a type of equations appear in several contexts in physics
- In particular, I will show later that it appears in black hole physics and in high energy physics

How?

- Mathematically, the spheroidal differential equation belongs to confluent Heun equations
- This has a nice connection with N=2 supersymmetric gauge theories
- I would like to explain what we can learn from this connection

Contents

◉ Review of spheroidal equations

◉ Relation to black hole physics

◉ Eigenvalues

◉ Relation to Nekrasov function

Spheroidal Coordinates

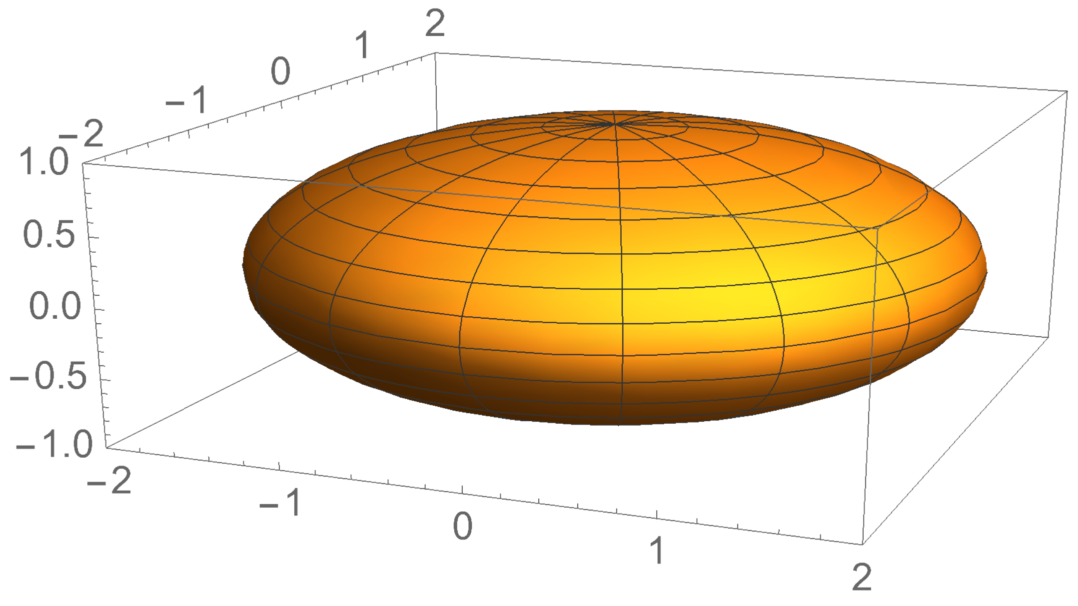
- ## Oblate coordinates

$$x = \alpha \cosh \mu \cos \nu \cos \phi$$

$$y = \alpha \cosh \mu \cos \nu \sin \phi$$

$$z = \alpha \sinh \mu \sin \nu$$

$$\frac{x^2 + y^2}{\alpha^2 \cosh^2 \mu} + \frac{z^2}{\alpha^2 \sinh^2 \mu} = 1$$



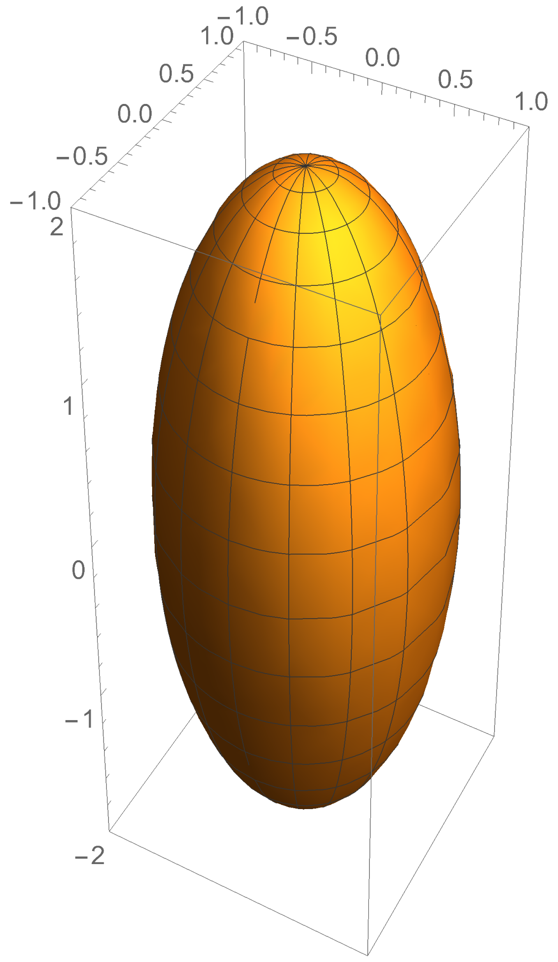
- ## Prolate coordinates

$$x = \alpha \sinh \mu \sin \nu \cos \phi$$

$$y = \alpha \sinh \mu \sin \nu \sin \phi$$

$$z = \alpha \cosh \mu \cos \nu$$

$$\frac{x^2 + y^2}{\alpha^2 \sinh^2 \mu} + \frac{z^2}{\alpha^2 \cosh^2 \mu} = 1$$



- **I focus on the oblate case**
- **It is useful to change variables**

$$\zeta := \sinh\mu, \quad \xi := \sin\nu$$

$$x = \alpha\sqrt{(1+\zeta^2)(1-\xi^2)}\cos\phi$$

$$y = \alpha\sqrt{(1+\zeta^2)(1-\xi^2)}\sin\phi$$

$$z = \alpha\zeta\xi$$

$$\begin{aligned} ds^2 &= dx^2 + dy^2 + dz^2 \\ &= \alpha^2(\zeta^2+\xi^2)\left(\frac{d\zeta^2}{1+\zeta^2} + \frac{d\xi^2}{1-\xi^2}\right) + \alpha^2(1+\zeta^2)(1-\xi^2)d\phi^2 \end{aligned}$$

$$ds^2 = \alpha^2(\zeta^2+\xi^2)\left(\frac{d\zeta^2}{1+\zeta^2}+\frac{d\xi^2}{1-\xi^2}\right)+\alpha^2(1+\zeta^2)(1-\xi^2)d\phi^2$$

$$\nabla^2 = \frac{1}{\alpha^2(\zeta^2+\xi^2)}\left[\frac{\partial}{\partial\zeta}(1+\zeta^2)\frac{\partial}{\partial\zeta}+\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi}\right]+\frac{1}{\alpha^2(1+\zeta^2)(1-\xi^2)}\frac{\partial^2}{\partial\phi^2}$$

- **Helmholtz equation:** $(\nabla^2+k^2)\Psi = 0$

$$\left[\frac{\partial}{\partial\zeta}(1+\zeta^2)\frac{\partial}{\partial\zeta}+\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi}+\frac{\zeta^2+\xi^2}{(1+\zeta^2)(1-\xi^2)}\frac{\partial^2}{\partial\phi^2}+\alpha^2k^2(\zeta^2+\xi^2)\right]\Psi = 0$$

- **This PDE is separable**

$$\Psi = R(\zeta)S(\xi)e^{im\phi}$$

- **Helmholtz equation:** $(\nabla^2 + k^2)\Psi = 0$

$$\left[\frac{\partial}{\partial\zeta}(1+\zeta^2)\frac{\partial}{\partial\zeta} + \frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi} + \frac{\zeta^2+\xi^2}{(1+\zeta^2)(1-\xi^2)}\frac{\partial^2}{\partial\phi^2} + \alpha^2 k^2(\zeta^2+\xi^2)\right]\Psi = 0$$

- **This PDE is separable**

$$\Psi = R(\zeta)S(\xi)e^{im\phi} \qquad (0 \le \zeta < +\infty,\ -1 \le \xi \le 1)$$

$$\left[\frac{d}{d\zeta}(1+\zeta^2)\frac{d}{d\zeta} + (\alpha k\zeta)^2 + \frac{m^2}{1+\zeta^2} - \lambda\right]R(\zeta) = 0$$ **"Radial"**

$$\left[\frac{d}{d\xi}(1-\xi^2)\frac{d}{d\xi} + (\alpha k\xi)^2 - \frac{m^2}{1-\xi^2} + \lambda\right]S(\xi) = 0$$ **"Angular"**

Relation to black hole physics

Kerr Black Holes

- **Now I show the same equation appear in black hole physics**
- **Kerr metric (rotating black holes)**

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2$$

$$+\left(r^2 + a^2 + \frac{2Ma^2 r\sin^2\theta}{\Sigma}\right)\sin^2\theta d\phi^2$$

$$\Delta = r^2 - 2Mr + a^2 \qquad \Sigma = r^2 + a^2\cos^2\theta$$

Scalar in Kerr BH

- Let us consider a (massless) scalar field in the Kerr geometry

$$(-g)^{-1/2}\partial_\mu[(-g)^{1/2}g^{\mu\nu}\partial_\nu]\Phi = 0$$

- The wave equation is very complicated, but it is separable!

Detweiler 1980

$$\Phi = R(r)S(\theta)e^{-i\omega t+im\phi}$$

Scalar in Kerr BH

$$\Phi = R(r)S(\theta)e^{-i\omega t + im\phi}$$

Detweiler 1980

$$\left[\Delta \frac{d}{dr}\Delta \frac{d}{dr} + (r^2 + a^2)^2\omega^2 - 4aMmr\omega + a^2m^2 - (a^2\omega^2 + \lambda)\Delta\right]R = 0$$

$$\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{d}{d\theta} + (a\omega)^2\cos^2\theta - \frac{m^2}{\sin^2\theta} + \lambda\right]S = 0$$

$$\downarrow \ \xi = \cos\theta$$

$$\left[\frac{d}{d\xi}(1-\xi^2)\frac{d}{d\xi} + (a\omega\xi)^2 - \frac{m^2}{1-\xi^2} + \lambda\right]S(\xi) = 0$$

Comments

- **The radial component looks more complicated than the “radial” spheroidal equation**
- **If one considers a spin-s field, one obtains the generalized spheroidal equation**

$$\left[\frac{d}{d\xi}(1-\xi^2)\frac{d}{d\xi}+(c\xi)^2-2cs\xi+s-\frac{(m+s\xi)^2}{1-\xi^2}+\lambda\right]S(\xi)=0$$

Eigenvalues

- **We consider an eigenvalue problem of the spheroidal equation**

$$\left[\frac{d}{d\xi}(1-\xi^2)\frac{d}{d\xi}+(c\xi)^2-\frac{m^2}{1-\xi^2}+\lambda\right]S(\xi)=0$$

- **Boundary condition:**

$$S(\xi)\ \text{ is regular at }\ \xi=\pm 1$$

- **Note: It can be evaluated by Mathematica**

`SpheroidalEigenvalue`

Eigenvalues

- **When c=0, the equation reduces to the associated Legendre equation (or the spherical harmonics)**

$$\left[\frac{d}{d\xi}(1-\xi^2)\frac{d}{d\xi} - \frac{m^2}{1-\xi^2} + \lambda\right]S(\xi) = 0$$

$$\lambda(c=0) = \ell(\ell+1)$$

$$S(c=0,\xi) = P_\ell^m(\xi)$$

Perturbative Series

- **We can regard the spheroidal equation as a deformation of the associated Legendre equation**

$$\lambda_{\ell m} = \ell(\ell+1) + \sum_{k=1}^{\infty} c^{2k} \lambda_{\ell m}^{(k)}$$

$$S_{\ell m}(\xi) = P_{\ell}^{m}(\xi) + \sum_{k=1}^{\infty} c^{2k} S_{\ell m}^{(k)}(\xi)$$

Perturbative Series

- **A perturbative technique a la Bender and Wu works**

$$\lambda_{00} = -\frac{c^2}{3} - \frac{2c^4}{135} - \frac{4c^6}{8505} + \frac{26c^8}{1913625} + \mathcal{O}(c^{10})$$

$$S_{00}(x) = 1 + \frac{c^2x^2}{6} + c^4\left(\frac{x^4}{120} + \frac{x^2}{135}\right) + c^6\left(\frac{x^6}{5040} + \frac{x^4}{1890} + \frac{2x^2}{8505}\right)$$
$$+c^8\left(\frac{x^8}{362880} + \frac{x^6}{68040} + \frac{4x^4}{212625} - \frac{13x^2}{1913625}\right) + \mathcal{O}(c^{10})$$

- **Another efficient way is to use continued fractions**

C. Flammer, "Spheroidal Wave Functions"

Perturbative Series

- **This small-c expansion has a finite radius of convergence**

(ℓ, m)	Radii
(0, 0)	3.18
(2, 0)	3.18
(2, ±1)	5.93
(2, ±2)	5.85

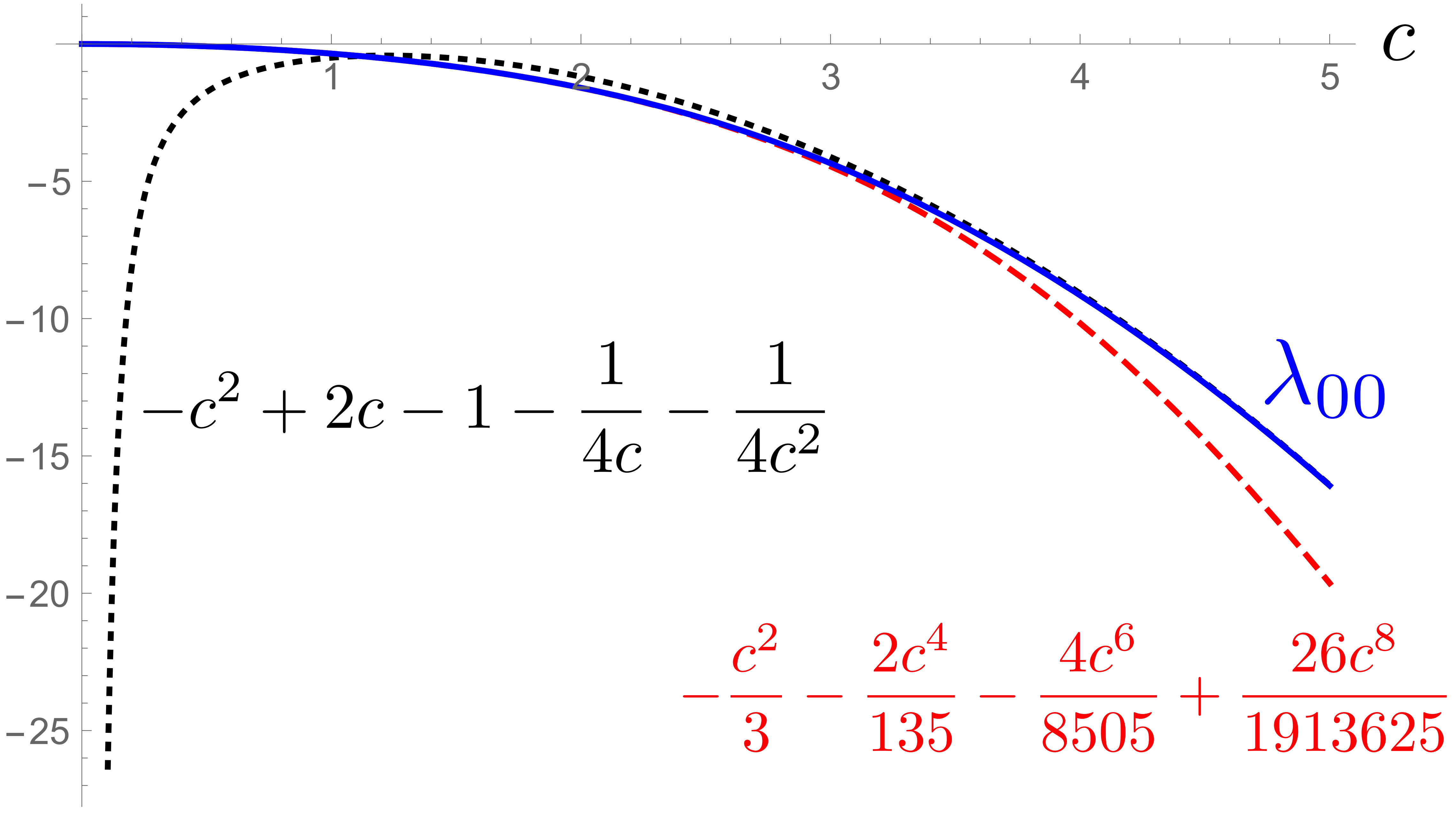
(ℓ, m)	Radii
(1, 0)	4.59
(1, ±1)	4.56
(3, 0)	4.59
(3, ±1)	4.56
(3, ±2)	7.21
(3, ±3)	7.10

Large-c Expansions

- **The large-c expansion seems to be divergent**

$$\lambda_{\ell m} \simeq -c^2 + \sum_{k=-1}^{\infty} \frac{\Lambda_{\ell m}^{(k)}}{c^k}$$

- **I have not yet understood its Stokes phenomenon etc...**



Relation to Nekrasov function

Confluent Heun Equation

- **The spheroidal equation has the same singularity structure as the confluent Heun equation**

$$\left[\frac{d}{d\xi}(1-\xi^2)\frac{d}{d\xi}+(c\xi)^2-\frac{m^2}{1-\xi^2}+\lambda\right]S(\xi)=0$$

$\xi = \pm 1$ **Regular singular points**

$\xi = \infty$ **Irregular singular point**

Confluent Heun Equation

- **We can rewrite it as the normal form of the confluent Heun equation**

$$\left(\frac{d^2}{dz^2} + Q_{\mathrm{SH}}(z)\right)\Psi(z) = 0$$

$$Q_{\mathrm{SH}}(z) = \frac{1}{z^2(z-1)^2}\sum_{i=0}^{4} A_i(\lambda, m, c) z^i$$

$z = 0, 1$ **Regular singular points**

$z = \infty$ **Irregular singular point**

Confluent Heun Equation

- **The same type of ODEs appears in the N=2 SU(2) SQCD with three fundamental hypermultiplets**

Gaiotto 2009

- **Seiberg-Witten curve:**

Hanany & Oz 1995

$$K(p) - \frac{\Lambda^{1/2}}{2}(e^{ix/2}K_+(p)e^{ix/2} + e^{-ix/2}K_-(p)e^{-ix/2}) = 0$$

$$K(p) = p^2 - u + \frac{\Lambda}{4}\left(p + \frac{m_1 + m_2 + m_3}{2}\right)$$

$$K_+(p) = (p + m_1)(p + m_2)$$

$$K_-(p) = p + m_3$$

Quantum SW Curve

- **We regard x and p as canonical variables, and replace them by noncommuting operators**

$$[x, p] = i\,, \qquad\qquad p = -i\partial_x$$

SW curve → 2nd order ODE

- **This ODE is equivalent to the confluent Heun equation**

Quantum SW Curve

Zenkevich 2012; Ito, Kanno & Okubo 2017

$$\left[K(-i\partial_x) - \frac{\Lambda^{1/2}}{2}\left(e^{ix/2}K_+(-i\partial_x)e^{ix/2} + e^{-ix/2}K_-(-i\partial_x)e^{-ix/2}\right)\right]\Psi(x) = 0$$

$$\Big\downarrow\; z = \frac{2}{\sqrt{\Lambda}}e^{-ix}$$

$$\left(\frac{d^2}{dz^2} + Q_{\mathrm{SW}}(z)\right)\psi(z) = 0$$

$$Q_{\mathrm{SW}}(z) = \frac{1}{z^2(z-1)^2}\sum_{i=0}^{4}\widetilde{A}_i(u,\vec{m},\Lambda)z^i$$

Comparison

- **Now we compare these two equations**

Aminov, Grassi & YH 2020

$$\Lambda = 16c, \qquad u = \lambda + c^2 + \frac{1}{4}$$

$$m_1 = -m, \qquad m_2 = m_3 = 0$$

- **This identification finally leads to an exact relation to the Nekrasov function**

Relation to Nekrasov Function

- **The Nekrasov function is a generalization of the Seiberg-Witten prepotential**
- **It exactly describes low-energy physics of N=2 theories**
- **Let us define**

$$F_{\mathrm{NS}}(a, \vec{m}, \Lambda) := \lim_{\epsilon_2 \to 0} \epsilon_1 \epsilon_2 \log Z_{\mathrm{Nekrasov}}(a, \vec{m}, \Lambda; \epsilon_1, \epsilon_2)|_{\epsilon_1 = 1}$$

Relation to Nekrasov Function

- **Matone relation:**

$$u = a^2 + \Lambda \frac{\partial F_{\rm NS}(a, \vec{m}, \Lambda)}{\partial \Lambda}$$

Matone 1995

- **Quantization condition (conjecture):**

$$a = \ell + \frac{1}{2}$$

Aminov, Grassi & YH 2020

- **The eigenvalue is finally given by**

$$\lambda_{\ell m}(c) = \ell(\ell+1) - c^2 + \Lambda \frac{\partial F_{\rm NS}}{\partial \Lambda}\Big|_{a=\ell+\frac{1}{2}, \vec{m}=(-m,0,0), \Lambda=16c}$$

Final Comments

- The small-c expansion corresponds to the “instanton expansion” of the Nekrasov function
- This can be checked directly
- On the other hand, in the large-Λ regime, the Nekrasov function has not been understood well

Summary

- **We found**

  (Spheroidal Eigenvalue) $\subset$ (Nekrasov Function)

- **The large-c expansion is probably useful to understand the large-Λ limit of the Nekrasov function**

- **The generalization to the spin-s case is straightforward**