Black hole microstate counting & Picard-Lefschetz theory

Masazumi Honda

(本多正純)

YUKAWA INSTITUTE FOR THEORETICAL PHYSICS

CGP Center for Gravitational Physics Yukawa Institute for Theoretical Physics

Kyoto University

Ref: MH, work in progress

Related works:

Choi-Kim-Kim-Nahmgoong, arXiv:1810.12067; MH, arXiv:1901.08091; Ardehali, arXiv:1902.06619
Benini-Milan, arXiv:1812.09613; Cabo-Bizet-Murthy, arXiv:1909.09597;
Copetti-Grassi-Komargodski-Tizzano, arXiv:2008.04950; Ardehali-Hong-Liu, arXiv:1912.04169, etc…

25th, Sep., 2020

Potential Toolkit to Attack Nonperturbative Aspects of QFT
- Resurgence and related topics - @YITP, Kyoto

[Bekenstein ’72, Hawking ’75]

Black hole = Thermodynamic?

Entropy: $$\boxed{S_{\mathrm{BH}} = \frac{\mathrm{Area}}{4G}}$$

[Event Horizon Telescope
https://www.eso.org/public/images/eso1907]

Test of Quantum Gravity

Bekenstein-Hawking entropy:

[Bekenstein '72, Hawking '75]

$$S_{\mathrm{BH}} = \frac{\mathrm{Area}}{4G}$$

macroscopic

What is a *microscopic* origin of the entropy?

Candidates of quantum gravity should answer

$\ni$ *String Theory!*

Answer from string theory for a specific black hole

[Strominger-Vafa '96]

For an asymptotically flat supersymmetric (SUSY) charged BH,

$$S_{BH} = \log \left(\begin{array}{l} \text{\# of states in string theory} \\ \text{w/ the same quantum numbers} \end{array} \right)$$

statistical explanation of BH entropy!

Another challenge for asymptotically AdS BH

AdS/CFT correspondence:

[Maldacena '97]

d-dim. CFT $\overset{\textit{dual}}{\Longleftrightarrow}$ Gravity in $\mathrm{AdS}_{\mathrm{d+1}}$

Does the CFT side give a microscopic explanation of BH entropy on the gravity side?

Another challenge for asymptotically AdS BH

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[Maldacena '97]

$$\text{d-dim. CFT} \quad \overset{\textit{dual}}{\Longleftrightarrow} \quad \text{Gravity in } \mathrm{AdS}_{d+1}$$

Does the CFT side give a microscopic explanation of BH entropy on the gravity side?

A natural expectation:

$$S_{BH} = \log\left(\begin{array}{l} \text{\# of states in CFT} \\ \text{w/ the same quantum numbers} \end{array} \right) \quad ??$$

This talk

Set up:

4d $SU(N)$ $\mathcal{N} = 4$ Super Yang-Mills theory $\overset{G_N \sim \frac{1}{N^2}}{\Longleftrightarrow}$ IIB string on $AdS_5 \times S^5$

This talk

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Super Yang-Mills theory $\overset{G_N \sim \frac{1}{N^2}}{\Longleftrightarrow}$ IIB string on $AdS_5 \times S^5$

$\exists$ black holes on the string side (rotating charged): [Gutowski-Reall '04 etc.]

$$S_{\mathrm{BH}} = \frac{A}{4G} = 2\pi\sqrt{Q_1Q_2 + Q_1Q_3 + Q_2Q_3 - \frac{N^2}{2}(J_1 + J_2)}$$

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Recent progress: (explained later)

[Choi-Kim-Kim-Nahmgoong, Benini-Milan, MH, Ardehali, Cabo-Bizet-Murthy, etc.]

The BH entropy is captured by superconformal index

but $^\exists$ subtle points w/ smells of Stokes phenomena

Resurgence can help?

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Related aspects:

- (de)confinement transition [Copetti-Grassi-Komargodski-Tizzano] ($\Leftrightarrow$ Hawking-Page transition)
- partial deconfinement [Ardehali-Hong-Liu]

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2. Recent progress & Subtle points

3. Resurgence to the rescue (?)

4. Summary

Black holes on the gravity side

4d $SU(N)$ $\mathcal{N}=4$ SYM $\longleftrightarrow$ IIB string on $AdS_5 \times S^5$

∃ Black holes on the string side w/

- 2 angular momenta: (J_1, J_2)
- 3 electric charges: (Q_1, Q_2, Q_3)
- 2 supercharges (1/16 BPS)
- mass: $g(|J_1| + |J_2| + |Q_1| + |Q_2| + |Q_3|)$

[Gutowski-Reall ’04 etc.]

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[Gutowski-Reall '04 etc.]

Bekenstein-Hawking entropy:

$$S_{\rm BH} = 2\pi\sqrt{Q_1Q_2 + Q_1Q_3 + Q_2Q_3 - \frac{\pi}{4G_N g^3}(J_1 + J_2)} \quad \text{with} \quad \frac{\pi}{2G_N g^3} = N^2$$

Is the entropy realized by counting CFT states?

Grand canonical partition function & Index

Q. Black hole entropy from counting CFT states?

Grand canonical partition function:

$$\mathrm{Tr}_{\mathrm{BPS}}\left[\prod_i x_i^{J_i}\prod_a y_a^{Q_a}\right]=\sum_{J,Q} d(Q,J)\prod_i x_i^{J_i}\prod_a y_a^{Q_a}$$

AdS/CFT→ $d(Q,J)|_{N,g_{\mathrm{YM}}^2 N\gg 1}\sim e^{S_{\mathrm{BH}}}=e^{\mathcal{O}(N^2)}$

difficult to compute

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Superconfomal index:

[Kinney-Maldacena-Minwalla-Raju'06]

$$I = \mathrm{Tr}_{\mathrm{BPS}}\left[(-1)^F\prod_i x_i^{J_i}\prod_a y_a^{Q_a}\right] = \sum_{J,Q}\left(d_{\mathrm{Bos}}(Q,J) - d_{\mathrm{Fer}}(Q,J)\right)\prod_i x_i^{J_i}\prod_a y_a^{Q_a}$$

$$d(Q,J) \geq d_{\mathrm{Bos}}(Q,J) - d_{\mathrm{Fer}}(Q,J)$$

under control

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Superconformal index in SU(N) N=4 SYM

In 4d $\mathcal{N}=1$ language,

4d $\mathcal{N}=4$ SYM = 1 vector + 3 adjoint chirals $\Phi_{1,2,3}$

Partition function on $S^1_\beta \times M_3$ $(M_3 \sim S^3)$:

$$Z_{S^1_\beta \times S^3} = e^{-\beta E} I, \qquad I = \text{Tr}_{\text{BPS}} \left[(-1)^F p^{J_1 + \frac{r}{2}} q^{J_2 + \frac{r}{2}} v_1^{q_1} v_2^{q_2} \right]$$

- $J_{1,2}$: angular momenta along S^3
- $Q_{1,2,3}/2$: charges of $U(1)^3 \subset SO(6)_R$
- $r = \frac{2}{3}(Q_1 + Q_2 + Q_3)$: $U(1)_R$ charge
- $q_{1,2} = Q_{1,2} - Q_3$: $U(1)_{1,2}$ charge

	Φ_1	Φ_2	Φ_3
$U(1)_1$	$+1$	0	-1
$U(1)_2$	0	$+1$	-1

Mathematical description: finite dim. integral

It is known that superconformal index is 1-loop exact:

$$I = \frac{(p;p)^N (q;q)^N}{N!} \int_{-\frac{1}{2}}^{\frac{1}{2}} d^{N-1}x \prod_{i\neq j} \Gamma_e(x_{ij}+\sigma+\tau;\sigma,\tau) \ \prod_{i,j} \Gamma_e\left(x_{ij}+m_1+\frac{1}{3}(\sigma+\tau);\sigma,\tau\right)$$

$$\times \Gamma_e\left(x_{ij}+m_2+\frac{1}{3}(\sigma+\tau);\sigma,\tau\right) \Gamma_e\left(x_{ij}-m_1-m_2+\frac{1}{3}(\sigma+\tau);\sigma,\tau\right)$$

$e^{2\pi i x_j}$: holonomy along S^1, $x_{ij} = x_i - x_j$

m_1, m_2: chemical potential for $U(1)$ flavor symmetries

$p = e^{2\pi i\sigma}$, q$= e^{2\pi i\tau}$, $(a;q) = \prod_{k=0}^{\infty}(1-aq^k)$,

$$\Gamma_e(x;\sigma,\tau) = \prod_{j,k\geq 0} \frac{1-e^{-2\pi i x}p^{j+1}q^{k+1}}{1-e^{2\pi i x}p^j q^k}$$

“Lore” on superconformal index until 2018

[Kinney-Maldacena-Minwalla-Raju’06, etc.]

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Let us consider

$(p = e^{2\pi i \sigma},\ \ \mathrm{q} = e^{2\pi i \tau})$

$$N \to \infty,\ m_1 = m_2 = 0,\ 0 < p, q < 1$$

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Let us consider

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Then, the famous result:

$$I = e^{\mathcal{O}(1)}$$

independent of N!?

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$$N \to \infty,\ m_1 = m_2 = 0,\ 0 < p, q < 1$$

Then, the famous result:

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Obviously, this does not capture the BH entropy:

$$S_{\mathrm{BH}} = \frac{A}{4G} = 2\pi\sqrt{Q_1Q_2 + Q_1Q_3 + Q_2Q_3 - \frac{N^2}{2}(J_1+J_2)}$$

Standard interpretation was

$\exists$ many cancelations due to $(-1)^F$

Loophole of the "lore"

[Choi-Kim-Kim-Nahmgoong, Cabo-Bizet-Cassani-Martelli-Murthy, etc.]

$$I = \mathrm{Tr}_{\mathrm{BPS}}\left[(-1)^F p^{J_1+\frac{r}{2}} q^{J_2+\frac{r}{2}} v_1^{q_1} v_2^{q_2}\right]$$

To pick up # of states, we have to extract coefficients:

$$(\text{\# of states}) = \oint \frac{dp}{p^{N_P+1}} \frac{dq}{q^{N_q+1}} \frac{dv_1}{v_1^{N_{v_1}+1}} \frac{dv_2}{v_2^{N_{v_2}+1}} I(p,q,v_1,v_2)$$

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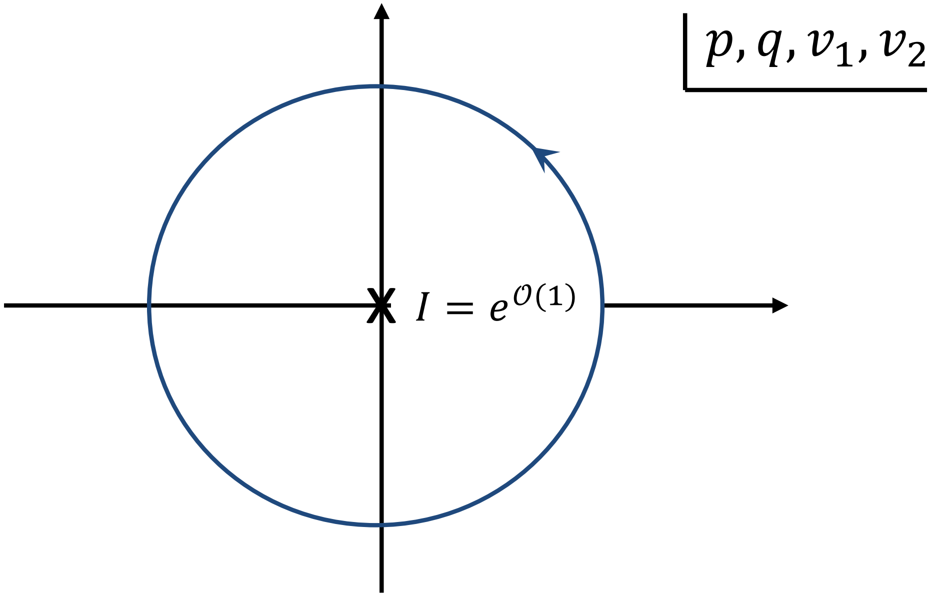

Loophole of the "lore"

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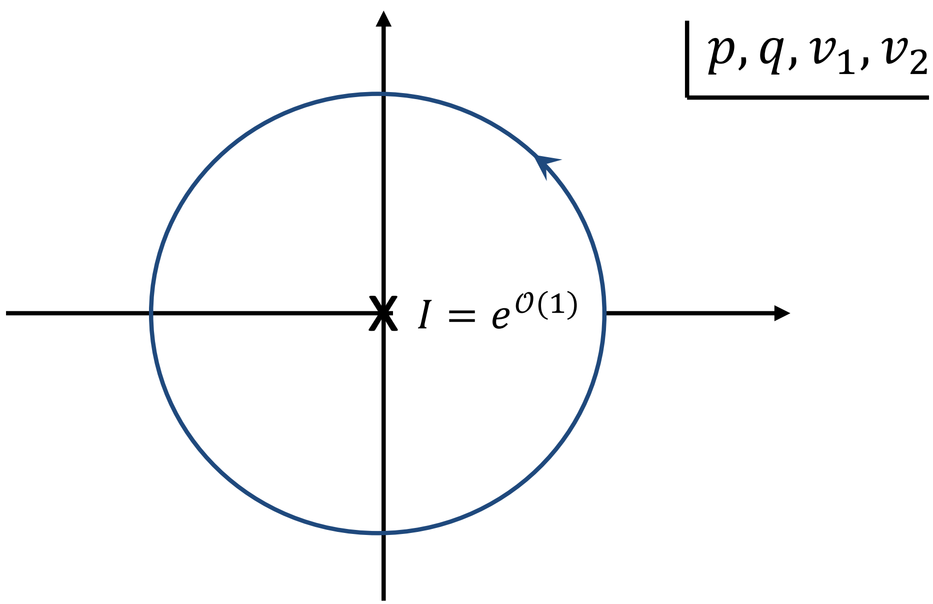

Need information for complex fugacities!

Recent updates

(details explained in next pages)

[Choi-Kim-Kim-Nahmgoong, Benini-Milan, MH, Ardehali, Cabo-Bizet-Murthy, etc.]

Loophole: complex fugacities may give different results

In the shrinking limit of S^1 ("Cardy limit"),

$$I = e^{\mathcal{O}(N^2)}$$ (for complex fugacities)

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Loophole: complex fugacities may give different results

In the shrinking limit of S^1 ("Cardy limit"),

$$I = e^{\mathcal{O}(N^2)}$$

(for complex fugacities)

Picking up (# of states) from this formula w/ $N \to \infty$ gives

$$S_{\text{BH}} = \frac{A}{4G} = 2\pi\sqrt{Q_1Q_2 + Q_1Q_3 + Q_2Q_3 - \frac{N^2}{2}(J_1 + J_2)}$$

SUSY Cardy formula for N=4 SYM

$$I_{S^1_\beta \times S^3} = \mathrm{Tr}_{\mathrm{BPS}}\left[(-1)^F p^{J_1+\frac{r}{2}} q^{J_2+\frac{r}{2}} v_1^{q_1} v_2^{q_2}\right]$$

[Di Pietro-Komargodski, Ardehali '15,Di Pietro-MH '16]

$$p = e^{2\pi i\sigma} = e^{\mathcal{O}(\beta)},\ q = e^{2\pi i\tau} = e^{\mathcal{O}(\beta)},\ v_{1,2} = e^{2\pi i m_{1,2}}$$

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Cardy limit $\beta \to 0$ $(\tau, \sigma \to 0)$:

$$I_{S^1\times S^3} \to \int_{-1/2}^{1/2} d^{N-1}a\ e^{\frac{i\pi}{6\tau\sigma}V_2(a) + \frac{i\pi(\tau+\sigma)}{2\tau\sigma}V_1(a)}$$

$$\left\{\begin{aligned}
V_2(a) &= -\sum_{1\le i\ne j\le N}\Big[\kappa(a_{ij}+m_1)+\kappa(a_{ij}+m_2)+\kappa(a_{ij}-m_1-m_2)\Big] \\
&\quad -(N-1)\Big[\kappa(m_1)+\kappa(m_2)+\kappa(-m_1-m_2)\Big], \\
V_1(a) &= \frac{1}{3}\sum_{1\le i\ne j\le N}\Big[3\theta(a_{ij})-\theta(a_{ij}+m_1)-\theta(a_{ij}+m_2)-\theta(a_{ij}-m_1-m_2)\Big] \\
&\quad -\frac{N-1}{3}\Big[\theta(m_1)+\theta(m_2)+\theta(-m_1-m_2)\Big], \\
a_{ij} &= a_i - a_j, \quad \sum_{j=1}^N a_j = 0, \qquad \kappa(x) = \{x\}(1-\{x\})(1-2\{x\}), \quad \theta(x) = \{x\}(1-\{x\}),
\end{aligned}\right.$$

Effective potential analysis

[MH '19]

$$I_{S^1\times S^3} \to \int_{-1/2}^{1/2} d^{N-1}a\, e^{\frac{i\pi}{6\tau\sigma}V_2(a)+\frac{i\pi(\tau+\sigma)}{2\tau\sigma}V_1(a)}$$

Let's focus on the regime $\mathrm{Re}\left(\frac{i}{\tau\sigma}\right) < 0$ → minimize $V_2(a)$!

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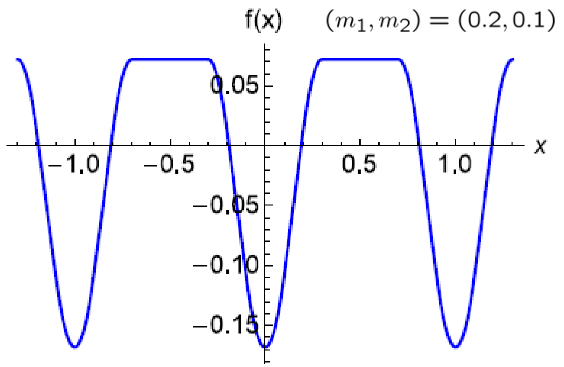
Let's focus on the regime $\mathrm{Re}\left(\frac{i}{\tau\sigma}\right) < 0$ → minimize $V_2(a)$!

$$\begin{cases} V_2(a) = \sum_{i<j} f(a_{ij}) + \frac{N-1}{2} f(0), \\ f(a_{ij}) \;=\; \kappa(a_{ij}-\{m_1\}) - \kappa(a_{ij}+\{m_1\}) + \kappa(a_{ij}-\{m_2\}) - \kappa(a_{ij}+\{m_2\} \\ \qquad +\kappa(a_{ij}+\{m_1\}+\{m_2\}) - \kappa(a_{ij}-\{m_1\}-\{m_2\}). \end{cases}$$



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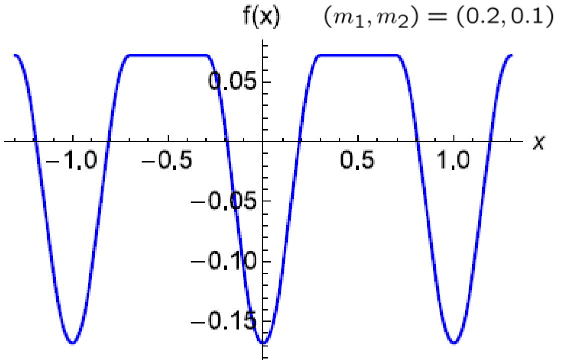
[MH '19]

Let's focus on the regime $\mathrm{Re}\left(\frac{i}{\tau\sigma}\right) < 0$ → minimize $V_2(a)$!

$$\begin{cases} V_2(a) = \sum_{i<j} f(a_{ij}) + \frac{N-1}{2} f(0), \\ f(a_{ij}) = \kappa(a_{ij}-\{m_1\}) - \kappa(a_{ij}+\{m_1\}) + \kappa(a_{ij}-\{m_2\}) - \kappa(a_{ij}+\{m_2\}) \\ \qquad +\kappa(a_{ij}+\{m_1\}+\{m_2\}) - \kappa(a_{ij}-\{m_1\}-\{m_2\}). \end{cases}$$



$f(x)$ has the minimum at the origin:

$$f(x)|_{\min} = f(0) = 12\{m_1\}\{m_2\}(\{m_1\}+\{m_2\}-1).$$

Thus,

$$\log I_{S^1\times S^3} \underset{|\tau|,|\sigma|\to 0}{\simeq} \frac{i\pi(N^2-1)}{\tau\sigma}\Big[\{m_1\}\{m_2\}(\{m_1\}+\{m_2\}-1) + \frac{\tau+\sigma}{3}\left(\{m_1\}^2+\{m_2\}^2+\{m_1\}\{m_2\}-\{m_1\}-\{m_2\}\right)\Big].$$

Entropy from index

$$I_{S^1\times S^3} = \text{Tr}_{\text{BPS}}\left[(-1)^F p^{J_1+\frac{r}{2}} q^{J_2+\frac{r}{2}} v_1^{q_1} v_2^{q_2}\right] \qquad (p=e^{2\pi i\sigma},\ q=e^{2\pi i\tau},\ v_{1,2}=e^{2\pi i m_{1,2}})$$

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Step 1: Redefine chemical potentials: $m_{1,2} = \Delta_{1,2} - \frac{\tau+\sigma}{3}$,

$$I_{S^1\times S^3} = \text{Tr}_{\text{BPS}}\left[(-1)^F p^{J_1+Q_3} q^{J_2+Q_3} e^{2\pi i\Delta_1(Q_1-Q_3)} e^{2\pi i\Delta_2(Q_2-Q_3)}\right]$$

$$= \text{Tr}_{\text{BPS}}\left[p^{J_1} q^{J_2} \prod_{a=1}^{3} e^{2\pi i\Delta_a Q_a}\right]_{\sum_a \Delta_a-\tau-\sigma-1\in 2\mathbf{Z}} \qquad \left((-1)^F = e^{2\pi i Q_3}\right)$$

[cf. Cabo-Bizet-Cassani-Martelli-Murthy '18]

$$\log I_{S^1\times S^3} \underset{|\tau|,|\sigma|\to 0}{\simeq} \frac{i\pi(N^2-1)\{\Delta_1\}\{\Delta_2\}(\{\Delta_1\}+\{\Delta_2\}-1-\sigma-\tau)}{\tau\sigma}$$

Entropy from index

$$I_{S^1\times S^3} = \mathrm{Tr}_{\mathrm{BPS}}\left[(-1)^F p^{J_1+\frac{r}{2}} q^{J_2+\frac{r}{2}} v_1^{q_1} v_2^{q_2}\right] \qquad (p = e^{2\pi i\sigma},\ q = e^{2\pi i\tau},\ v_{1,2} = e^{2\pi i m_{1,2}})$$

$$\log I_{S^1\times S^3} \underset{|\tau|,|\sigma|\to 0}{\simeq} \frac{i\pi(N^2-1)}{\tau\sigma}\Big[\{m_1\}\{m_2\}(\{m_1\}+\{m_2\}-1) + \frac{\tau+\sigma}{3}\left(\{m_1\}^2+\{m_2\}^2+\{m_1\}\{m_2\}-\{m_1\}-\{m_2\}\right)\Big].$$

Step 1: Redefine chemical potentials: $m_{1,2} = \Delta_{1,2} - \frac{\tau+\sigma}{3}$,

$$I_{S^1\times S^3} = \mathrm{Tr}_{\mathrm{BPS}}\left[(-1)^F p^{J_1+Q_3} q^{J_2+Q_3} e^{2\pi i\Delta_1(Q_1-Q_3)} e^{2\pi i\Delta_2(Q_2-Q_3)}\right]$$

$$= \mathrm{Tr}_{\mathrm{BPS}}\left[p^{J_1} q^{J_2} \prod_{a=1}^{3} e^{2\pi i\Delta_a Q_a}\right]_{\sum_a \Delta_a-\tau-\sigma-1\in 2\mathbf{Z}} \qquad \left((-1)^F = e^{2\pi i Q_3}\right)$$

[cf. Cabo-Bizet-Cassani-Martelli-Murthy '18]

$$\log I_{S^1\times S^3} \underset{|\tau|,|\sigma|\to 0}{\simeq} \frac{i\pi(N^2-1)\{\Delta_1\}\{\Delta_2\}(\{\Delta_1\}+\{\Delta_2\}-1-\sigma-\tau)}{\tau\sigma}$$

Step 2: Pick up "degeneracy" via Legendre transformation

Entropy from index (cont'd)

$$I = \mathrm{Tr}_{\mathrm{BPS}} \left[p^{J_1} q^{J_2} \prod_{a=1}^{3} e^{2\pi i \Delta_a Q_a} \right]_{\sum_a \Delta_a - \tau - \sigma - 1 \in 2\mathbf{Z}}$$

Pick up "degeneracy":

$$\int d\sigma d\tau d\Delta_1 d\Delta_2 d\Delta_3 d\Lambda \; I e^{-2\pi i (J_1 \sigma + J_2 \tau + \sum_a \Delta_a Q_a)} e^{-2\pi i \Lambda (\sum_a \Delta_a - \tau - \sigma - 1)}$$

Entropy from index (cont'd)

$$I = \mathrm{Tr}_{\mathrm{BPS}}\left[p^{J_1}q^{J_2}\prod_{a=1}^{3}e^{2\pi i\Delta_a Q_a}\right]_{\sum_a \Delta_a-\tau-\sigma-1\in 2\mathbf{Z}}$$

Pick up "degeneracy":

$$\int d\sigma d\tau d\Delta_1 d\Delta_2 d\Delta_3 d\Lambda\ I e^{-2\pi i(J_1\sigma+J_2\tau+\sum_a \Delta_a Q_a)}e^{-2\pi i\Lambda(\sum_a \Delta_a-\tau-\sigma-1)}$$

Entropy w/ large quantum numbers:

$$S_{\mathrm{CFT}}(Q,J) = -\log I + 2\pi i(J_1\sigma + J_2\tau + \sum_a \Delta_a Q_a) + 2\pi i\Lambda(\sum_a \Delta_a - \tau - \sigma - 1)\Big|_{\mathrm{ext.}}$$

$$\Rightarrow \quad \boxed{S_{\mathrm{CFT}} = 2\pi\sqrt{Q_1Q_2+Q_1Q_3+Q_2Q_3-\frac{N^2-1}{2}(J_1+J_2)}}$$

Entropy from index (cont'd)

$$I = \mathrm{Tr}_{\mathrm{BPS}}\left[p^{J_1} q^{J_2} \prod_{a=1}^{3} e^{2\pi i \Delta_a Q_a}\right]_{\sum_a \Delta_a - \tau - \sigma - 1 \in 2\mathbf{Z}}$$

Pick up "degeneracy":

$$\int d\sigma d\tau d\Delta_1 d\Delta_2 d\Delta_3 d\Lambda \; I e^{-2\pi i (J_1\sigma + J_2\tau + \sum_a \Delta_a Q_a)} e^{-2\pi i \Lambda (\sum_a \Delta_a - \tau - \sigma - 1)}$$

Entropy w/ large quantum numbers:

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$$\Rightarrow \quad \boxed{S_{\mathrm{CFT}} = 2\pi \sqrt{Q_1 Q_2 + Q_1 Q_3 + Q_2 Q_3 - \frac{N^2 - 1}{2}(J_1 + J_2)}}$$

BH entropy:

$$\boxed{S_{\mathrm{BH}} = 2\pi \sqrt{Q_1 Q_2 + Q_1 Q_3 + Q_2 Q_3 - \frac{N^2}{2}(J_1 + J_2)}}$$

Agrees in the large-N limit!!

Quantum black hole entropy?

$$S_{\mathrm{CFT}} = 2\pi\sqrt{Q_1Q_2 + Q_1Q_3 + Q_2Q_3 - \frac{N^2-1}{2}(J_1+J_2)}$$

In terms of the central charge $c = \frac{N^2-1}{4}$, (True also for other gauge group and orbifold N=4 SYM)

$$S_{\mathrm{CFT}} = 2\pi\sqrt{Q_1Q_2 + Q_1Q_3 + Q_2Q_3 - 2c(J_1+J_2)}$$

Quantum black hole entropy?

$$S_{\mathrm{CFT}} = 2\pi\sqrt{Q_1Q_2 + Q_1Q_3 + Q_2Q_3 - \frac{N^2-1}{2}(J_1+J_2)}$$

In terms of the central charge $c = \frac{N^2-1}{4}$, (True also for other gauge group and orbifold N=4 SYM)

$$S_{\mathrm{CFT}} = 2\pi\sqrt{Q_1Q_2 + Q_1Q_3 + Q_2Q_3 - 2c(J_1+J_2)}$$

Somehow this is exactly the same as the BH formula

if we slightly modify the AdS/CFT dictionary:

$$\left.\frac{\pi}{2G_N g^3}\right|_{\mathrm{finite}N} = N^2 - 1 = 4c \quad \left(S_{\mathrm{BH}} = 2\pi\sqrt{Q_1Q_2 + Q_1Q_3 + Q_2Q_3 - \frac{\pi}{4G_N g^3}(J_1+J_2)} \right)$$

Non-renormalization against quantum corrections?

Other regime

$$I_{S^1\times S^3} \to \int_{-1/2}^{1/2} d^{N-1}a\; e^{\frac{i\pi}{6\tau\sigma}V_2(a)+\frac{i\pi(\tau+\sigma)}{2\tau\sigma}V_1(a)}$$

So far, $\mathrm{Re}\left(\frac{i}{\tau\sigma}\right)<0$. What if $\mathrm{Re}\left(\frac{i}{\tau\sigma}\right)>0$? $\to$ maximize $V_2(a)$

$$\begin{cases} V_2(a) = \sum_{i<j} f(a_{ij}) + \frac{N-1}{2} f(0), \\ f(a_{ij}) = \kappa(a_{ij}-\{m_1\}) - \kappa(a_{ij}+\{m_1\}) + \kappa(a_{ij}-\{m_2\}) - \kappa(a_{ij}+\{m_2\}) \\ \qquad +\kappa(a_{ij}+\{m_1\}+\{m_2\}) - \kappa(a_{ij}-\{m_1\}-\{m_2\}). \end{cases}$$

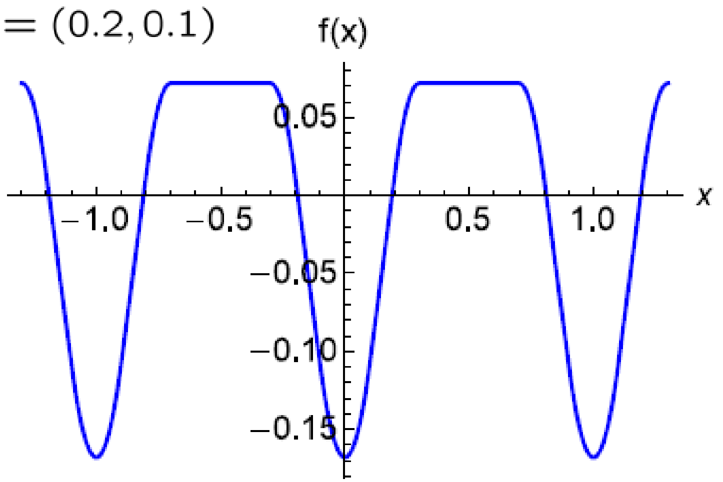
Other regime

$$I_{S^1\times S^3} \to \int_{-1/2}^{1/2} d^{N-1}a\ e^{\frac{i\pi}{6\tau\sigma}V_2(a)+\frac{i\pi(\tau+\sigma)}{2\tau\sigma}V_1(a)}$$

So far, $\mathrm{Re}\left(\frac{i}{\tau\sigma}\right)$<0. What if $\mathrm{Re}\left(\frac{i}{\tau\sigma}\right)$ >0 ? → maximize $V_2(a)$

$$\begin{cases} V_2(a) = \sum_{i<j} f(a_{ij}) + \frac{N-1}{2} f(0), \\ f(a_{ij}) = \kappa(a_{ij} - \{m_1\}) - \kappa(a_{ij} + \{m_1\}) + \kappa(a_{ij} - \{m_2\}) - \kappa(a_{ij} + \{m_2\}) \\ \qquad + \kappa(a_{ij} + \{m_1\} + \{m_2\}) - \kappa(a_{ij} - \{m_1\} - \{m_2\}). \end{cases}$$

$(m_1, m_2) = (0.2, 0.1)$



minima are not isolated
& ∃ flat directions

Stokes phenomena?

Subtleties

- In general, $^{\exists}$ complex saddle points so we need to do proper saddle point analysis

- But "action" in the strict Cardy limit is non-differentiable

  ($\because$ "action" is described by $\kappa(x) = \{x\}(1-\{x\})(1-2\{x\})$)

- It seems $^{\exists}$ Stokes phenomena as changing phases of chemical potentials

- Other works reported (de)confinement transition & partial deconfinement

[Copetti-Grassi-Komargodski-Tizzano]

[Ardehali-Hong-Liu]

Resurgence

Oops!

Next slides include preliminary results!