Perturbative ambiguities in compactified spacetime and resurgence structure

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- K. Ishikawa, O.M., K. Shibata and H. Suzuki, PTEP **2020** (2020) 063B02 [arXiv:2001.07302 [hep-th]]
- O.M. and H. Takaura, PLB **807** (2020) 135570 [arXiv:2003.04759 [hep-th]]
- M. Ashie, O.M., H. Suzuki and H. Takaura, PTEP **2020** (2020) 093B02 [arXiv:2005.07407 [hep-th]]

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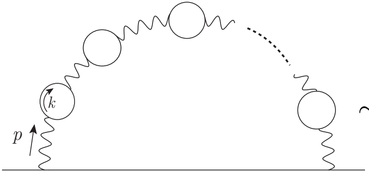
Factorial growth in QFT

- Two sources of factorial growth of perturbative coefficients:
  1. Proliferation of Feynman diagrams (PFD)

	Ground state energy in QM
Zeeman effect	$\sim (-1)^k (2k)!$
Stark effect	$\sim (2k)!$
$V(\phi) \sim \phi^3$	$\sim \Gamma(k+1/2)$
$V(\phi) \sim \phi^4$	$\sim (-1)^k \Gamma(k+1/2)$
Double well	$\sim k!$
Periodic cosine	$\sim k!$

  2. Renormalon ['t Hooft '79]



$\sim \beta_0^k k!$

(β_0: one-loop coefficient of the beta function)

Resurgence theory and renormalon problem

- Perturbative ambiguities (Borel resummation)

$$\text{PFD}: \quad \sum_k k!\left(g^2/16\pi^2\right)^k \qquad \Rightarrow \delta \sim \exp\left(-16\pi^2/g^2\right)$$

$$\text{Renormalon}: \quad \sum_k \beta_0^k k!\left(g^2/16\pi^2\right)^k \quad \Rightarrow \delta \sim \exp\left[-16\pi^2/(\beta_0 g^2)\right]$$

- Resurgence structure [Bogomolny '80, Zinn-Justin '81]

$$\delta_{\text{PFD}} \sim \exp\left(-16\pi^2/g^2\right) = \exp\left(-2S_I\right)$$

$\Updownarrow$ Cancellation

$$\delta_{\text{Instanton}} \sim \exp\left(-2S_I\right) \qquad (\text{Instanton action } S_I = 8\pi^2/g^2)$$

- But, what cancels the renormalon ambiguity $\delta_{\text{Renormalon}}$?
  - Non-trivial configuration with $S = S_I/\beta_0$ ($\delta_{\text{NP}} \sim e^{-2S_I/\beta_0}$)?
  - Not known (cf. cancellation between terms of OPE [David '82])
  - Higher-order perturbation theory?

Renormalon cancellation in $\mathbb{R}^{d-1} \times S^1$?

- A possible candidate [Argyres–Ünsal '12, Dunne–Ünsal '12, ...]:
  Bion [Ünsal '07] = fractional instanton/anti-instanton pair
  on $\mathbb{R}^{d-1} \times S^1$ with $\mathbb{Z}_N$-twisted boundary conditions (BC)

$$\underbrace{\varphi^A(\boldsymbol{x}, x_d + 2\pi R)}_{N\text{-component field}} = e^{2\pi i m_A R} \varphi^A(\boldsymbol{x}, x_d), \quad m_A R = \begin{cases} A/N & \text{for } A \neq N \\ 0 & \text{for } A = N \end{cases}$$

- $S_B = S_I / N \leftarrow$ similar N dependence! ($\beta_0 = \frac{11}{3} N$ for $SU(N)$ GT)
- Resurgence on S^1 compactified spacetime?

	PT	Semi-classical object
$\cancel{\mathbb{R}^4}$	$\delta_{\text{Renormalon}} \sim e^{-16\pi^2/(\beta_0 g^2)}$	?
$\mathbb{R}^3 \times S^1$	$\underline{\delta_{\text{Renormalon}}\,?}$	$\delta_{\text{Bion}} \sim e^{-2S_B} = e^{-2S_I/N}$

  - QFT on $\mathbb{R}^4 \overset{\text{Def.}}{=}$ QFT on $\mathbb{R}^3 \times S^1$ in $R \to \infty$

Resurgence structure in $\mathbb{R}^{d-1} \times S^1$

- *No renormalons* in $SU(2)$ and $SU(3)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$ [Anber–Sulejmanpasic '14]
- Questions:
  1. What cancels the bion ambiguity? (PFD?)
  2. Are there no renormalons in other theories? ($SU(^\forall N)$?)
  3. How does $\delta_{\text{Renormalon}}$ on $\mathbb{R}^4$ emerge in $R \to \infty$?
- Answers [O.M.–Takaura, Ashie–O.M.–Suzuki–Takaura]:
  1. Enhancement of PFD due to $\mathbb{Z}_N$ twisted BC: $k! \to N^k k!$

	PT	Semi-classical object
$\mathbb{R}^4$	$\delta_{\text{PFD}} \sim e^{-16\pi^2/g^2}$	$\delta_{\text{Instanton}} \sim e^{-2S_I}$
$\mathbb{R}^3 \times S^1$	$\delta'_{\text{PFD}} \sim e^{-16\pi^2/(Ng^2)}$	$\delta_{\text{Bion}} \sim e^{-2S_I/N}$

  2. No renormalons in $SU(^\forall N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$
  3. "Renormalon precursor" $\overset{R\to\infty}{\to}$ Renormalon on $\mathbb{R}^4$
- Helpful in giving a unified understanding on resurgence in QFT

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Enhancement of PFD ambiguity

- Enhanced PFD-type ambiguities cancel bion ambiguities

$$\text{PFD: } k! \Rightarrow N^k k!$$

$$\delta_{\text{PFD}} \sim e^{-16\pi^2/g^2} \Rightarrow \delta'_{\text{PFD}} \sim e^{-16\pi^2/(Ng^2)} \leftrightarrow \delta_{\text{bion}}\big|_{\mathbb{R}^3\times S^1}$$

- Mechanism ($\sim$ Linde problem ['80] in finite-temperature QFT):
  1. S^1 compactification $\to$ IR divergences

$$\int d^d p \cdots = \text{finite} \to \int d^{d-1} p \frac{1}{2\pi R} \sum_{p_d \in \mathbb{Z}/R} \cdots = \infty \text{ at IR}$$

  2. twisted BC $\to$ twist angles as IR regulators

$$\frac{1}{p^2} \to \frac{1}{\boldsymbol{p}^2 + (p_d + m_A)^2} \xrightarrow[p_d=0, A=1]{\text{Amplitude}} \left(\frac{g^2}{m_A R}\right)^k = \underbrace{(Ng^2)^k}_{\text{Enhancement!}}$$

- Let us study the $\mathbb{C}P^N$ model on $\mathbb{R} \times S^1$
  - Bion calculus [Fujimori-Kamata-Misumi-Nitta-Sakai '18]

2-dimensional $\mathbb{C}P^N$ model with twisted BC

- 2D $\mathbb{C}P^{N-1}$ model with $\mathbb{Z}_N$-twisted BC ($A = 1, 2, \ldots, N$)

$$S = \frac{1}{g^2} \int d^2x \left[\partial_\mu \bar{z}^A \partial_\mu z^A - j_\mu j_\mu + f(\bar{z}^A z^A - 1)\right]$$

where $j_\mu = (1/2i)\bar{z}^A \overleftrightarrow{\partial}_\mu z^A$, f is an auxiliary field ($\bar{z}^A z^A = 1$)

- # of vacuum bubble diagrams, T_k ($j_\mu \to \bar{z}z$, propagator $\to 1$)

$$T_k = \frac{1}{(2k)!} \left(\frac{\delta}{\delta \bar{z}} \frac{\delta}{\delta z}\right)^{2k} \frac{1}{k!} \left[(\bar{z}z)^2\right]^k \sim 4^k \Gamma(k + 1/2)$$

- Amplitude of $(k+1)$-loop connected diagram

$$\xrightarrow{p_2=0, A=1} \frac{V_2 (g^2)^k}{(2\pi R)^{k+1}} \int \left(\prod_{i=1}^{k+1} \frac{dp_{i,1}}{2\pi}\right) \frac{F^{2k}(p_{i,1}, m_A)}{\prod_{i=1}^{2k} [q_{i,1}^2 + m_A^2 + f_0]}$$

(F^{2k}: $2k$th-order polynomial, q: linear combination of $\{p_i\}$, $f_0 = \langle f \rangle$)

- IR divergence in massless limit $m_A^2 + f_0 \to 0$ ($\int d^2p \to \int dp$)

IR structure and enhancement mechanism

- $m_A^2 + f_0 = 1/(NR)^2 + f_0$ works as an IR regulator
  - $m_A^2 \gg f_0$

$$\sim \frac{V_2}{R^2}\frac{1}{(m_A R)^{k-1}}\left(\frac{g^2}{4\pi}\right)^k \xrightarrow{A=1} \frac{V_2}{R^2}\frac{1}{N}\left(\frac{Ng^2}{4\pi}\right)^k$$

Enhancement!

  - $m_A^2 \ll f_0$

$$\sim \frac{V_2}{R^2}\sum_{\alpha\geq 0}\frac{(m_A R)^\alpha}{(f_0 R^2)^{(k+\alpha-1)/2}}\left(\frac{g^2}{4\pi}\right)^k$$

~~Enhancement~~

- Dependence on $NR\Lambda$ ($\Lambda = \mu e^{-2\pi/(\beta_0 g^2)}$: dynamical scale)
  - $NR\Lambda \ll 1$ (Bion calculus is valid)

$$\sqrt{f_0}R \sim \frac{g^2}{4\pi} \Rightarrow [m_A^2 = \mathcal{O}(g^0)] \gg [f_0 = \mathcal{O}(g^4)]$$

Enhancement!

  - $NR\Lambda \gg 1$ ("Large N")

$$f_0 \sim \Lambda^2 \quad \Rightarrow \quad \frac{m_A^2}{f_0} = \frac{1}{(NR\Lambda)^2} \ll 1$$

~~Enhancement~~

Discussions on enhancement mechanism

	PT	Semi-classical object
$\mathbb{R}^4$	$\delta_{\text{PFD}} \sim e^{-16\pi^2/g^2}$	$\delta_{\text{Instanton}} \sim e^{-2S_I}$
$\mathbb{R}^3 \times S^1$	$\delta'_{\text{PFD}} \sim e^{-16\pi^2/(Ng^2)}$	$\delta_{\text{Bion}} \sim e^{-2S_I/N}$

- Enhancement phenomenon is consistent with bion
- Borel singularities at $m_A R = A/N = \underbrace{1/N}_{\text{1-bion}}, \underbrace{2/N}_{\text{2-bion}}, \ldots$
- Disconnected diagram (n connected) is suppressed by $1/N^{n-1}$
  - $\#$ of connected diagrams $\sim \ln \left[\sum_k T_k (g^2)^k\right]$
- Sum over flavor indices gives rise to further N factor
- Vacuum energy of 2D SUSY $\mathbb{C}P^N$ model (following slides)
  - $NR\Lambda \gg 1$ [Ishikawa–O.M.–Shibata–Suzuki]
  - $NR\Lambda \ll 1$ [Fujimori–Kamata–Misumi–Nitta–Sakai '18]

Vacuum energy of 2D SUSY $\mathbb{C}P^{N-1}$ model

- 2D SUSY $\mathbb{C}P^{N-1}$ model with SUSY breaking term

$$S = \frac{N}{\lambda}\int d^2x \left[-\bar{z}^A D_\mu D_\mu z^A + \bar{\sigma}\sigma + \bar{\chi}^A(\not{D} + \bar{\sigma}P_+ + \sigma P_-)\chi^A\right] + \int d^2x\, \frac{\delta\epsilon}{\pi R}\sum_A m_A \left(\bar{z}^A z^A - \frac{1}{N}\right)$$

where $D_\mu = \partial_\mu + iA_\mu$, $\gamma_5 = -i\gamma_x\gamma_y$, $P_\pm = (1 \pm \gamma_5)/2$, and we impose $\bar{z}^A z^A = 1$, $\bar{z}^A\chi^A = 0$ and $\bar{z}^A\chi^A = 0$.

- Vacuum energy as a function of $\delta\epsilon$

$$E(\delta\epsilon) = \underbrace{E^{(0)}}_{=0} + E^{(1)}\delta\epsilon + E^{(2)}\delta\epsilon^2 + \dots$$

- Vacuum energy at $NR\Lambda \ll 1$ [Fujimori-Kamata-Misumi-Nitta-Sakai]

$$E_{\mathrm{Bion}} = 2\Lambda \sum_{b=1}^{N-1}(-1)^b \frac{b}{(b!)^2}(NR\Lambda)^{2b-1} \times \left\{\delta\epsilon + \left[-2\gamma_E - 2\ln\left(4\pi b/\lambda_R\right) \mp \pi i\right]\delta\epsilon^2 + \dots\right\}$$

Vacuum energy in terms of $1/N$ expansion

- Vacuum energy ($NR\Lambda \gg 1$) [Ishikawa–O.M.–Shibata–Suzuki]

$$E^{(1)}\delta\epsilon = 2\delta\epsilon \sum_A m_A \left\langle \bar{z}^A z^A - 1/N \right\rangle_{\delta\epsilon=0}$$
$$= 0 \cdot N^0 + 0 \cdot N^{-1} + \mathcal{O}(N^{-2})$$

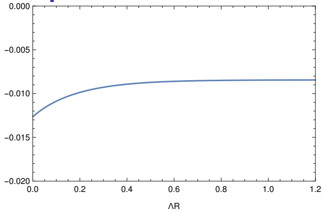

Figure: F

$$RE^{(2)}\delta\epsilon^2$$
$$= -\frac{\delta\epsilon^2}{\pi}\int d^2x \sum_{A,B} m_A m_B$$
$$\times \left\langle \bar{z}^A z^A(x) \bar{z}^B z^B(0) \right\rangle_{\delta\epsilon=0}$$
$$= N^{-1}(\lambda_R \delta\epsilon_R)^2 (\Lambda R)^{-2} F(\Lambda R)$$
$$+ N^{-2}(\lambda_R \delta\epsilon_R)^2 (\Lambda R)^{-3} G(\Lambda R) + O(N^{-3})$$

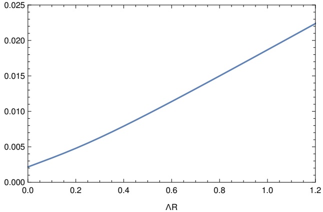

Figure: G

- $RE^{(2)} \to \infty$ as $\Lambda \to 0$; no well-defined weak coupling expansion

N-enhancement versus $1/\Lambda$ (m_A^2 vs f_0)

- Vacuum energy for $NR\Lambda \gg 1$ and $NR\Lambda \ll 1$

$$
\begin{array}{ccc}
E_{\text{Large }N} \sim N^{-1}(\Lambda R)^{-2}\delta\epsilon^2 & & \Rightarrow f_0 = \Lambda^2 \text{ in denominator} \\
\updownarrow & & \Downarrow (m_A^2 \text{ vs } f_0) \\
\operatorname{Im} E_{\text{1-bion}} \sim \pm N(\Lambda R)^2\delta\epsilon^2 & & \Leftarrow \text{enhancement of } N
\end{array}
$$

- cf. ground state energy of SUSY $\mathbb{C}P^1$ QM [Fujimori–Kamata–Misumi–Nitta–Sakai '17]

$$
E = m \sum_k A_k (g^2/m)^k
$$

- "Large N" means $NR\Lambda \gg 1$
  - "$1/N$ expansion" $\equiv 1/(NR\Lambda)$ expansion
  - Higher orders in $1/N$ depend on negative powers of Λ
  - $f_0 = \Lambda^2$ dominates the IR structure

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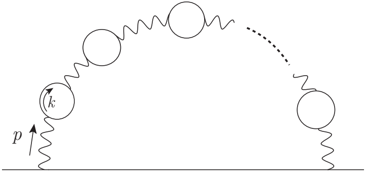
(Definition of) Renormalon ambiguity on $\mathbb{R}^4$

- Renormalon on $\mathbb{R}^4$ can appear from



$$\sim g^2(\mu^2) \sum_k \int \frac{d^4 p}{(2\pi)^4} (p^2)^\alpha \Pi(p^2)^k,$$

$$\Pi(p^2) = \beta_0 \frac{g^2(\mu^2)}{16\pi^2} \ln \frac{e^C \mu^2}{p^2}$$

(α, C: constants)

- As $p \to 0$, logarithmic factor in vacuum polarization is crucial

$$\int_p (p^2)^\alpha (\ln p^2)^k \overset{p\sim 0}{\to} k! \quad \overset{\text{Borel}}{\Rightarrow} \quad \pm i\pi \frac{1}{\beta_0} (e^C \Lambda^2)^{\alpha+2}$$

where dynamical scale $\Lambda^2 = \mu^2 e^{-16\pi^2/(\beta_0 g^2)}$

  - Borel transform $\to$ momentum integral

$$B(u) = \int d^4 p \, (p^2)^\alpha \left(\frac{e^C \mu^2}{p^2} \right)^u \xrightarrow{u \geq \alpha+2} \text{IR divergence}$$

$SU(N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$

- Renormalon analysis for $SU(N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$
  ($N = 2, 3$ [Anber-Sulejmanpasic '14], $\forall N$ [Ashie-O.M.-Suzuki-Takaura '20])
- $SU(N)$ gauge theory with n_W-flavor adjoint Weyl fermions

$$S = -\frac{1}{2g_0^2}\int d^4x \,\mathrm{tr}\left(F_{\mu\nu}F_{\mu\nu}\right) - 2\int d^4x \,\mathrm{tr}\,\bar{\psi}\gamma_\mu\left(\partial_\mu + [A_\mu, \psi]\right)$$

- $\mathbb{Z}_N$ twisted BC for adjoint representation

$$\psi(x_0, x_1, x_2, x_3 + 2\pi R) = \Omega\psi(x)\Omega^{-1}$$
$$A_\mu(x_0, x_1, x_2, x_3 + 2\pi R) = \Omega A_\mu(x)\Omega^{-1}$$

where $\Omega = e^{i\pi\frac{N+1}{N}}\,\mathrm{diag}(e^{-i\frac{2\pi}{N}}, e^{-i\frac{2\pi}{N}2}, \ldots, e^{-i\frac{2\pi}{N}N})$.

  - Equivalently, $e^{2\pi R A_3^{(0)}} = \Omega$ under periodic BC

- $\underbrace{\text{Cartan part}}_{N-1}$ of A_μ: $\underbrace{\text{“photon”}}_{\text{massless } U(1)}$, the others of A_μ: $\underbrace{\text{“W-boson”}}_{m\sim|p_3|\geq 1/(NR)}$
- W-boson cannot give rise to renormalon; consider $U(1)^{N-1}$

1-loop effective action

- 1-loop effective action of photon, A_μ^ℓ ($\ell = 1, 2, \ldots, N-1$)

$$\frac{1}{2g^2}\int d^4x d^4y\, A_\mu^\ell(x) A_\nu^r(y) \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\pi R}\sum_{p_3} e^{-ip(x-y)}$$
$$\times \left[p^2 \mathcal{P}^L_{\mu\nu}(\delta_{\ell r} - L_{\ell r}) + p^2 \mathcal{P}^T_{\mu\nu}(\delta_{\ell r} - T_{\ell r}) + \delta_{\ell r}\xi p_\mu p_\nu\right]$$

  where projection operators, $\mathcal{P}^L$ along S_1, $\mathcal{P}^T$ in $\mathbb{R}^3$

- Vacuum polarization

$$L_{\ell r} = \frac{\beta_0 g^2}{16\pi^2}\left[\delta_{\ell r}\ln\left(\frac{e^{5/3}\mu^2}{p^2}\right) + \underbrace{f_0(p^2R^2)_{\ell r} - f_2(p^2R^2)_{\ell r}}_{\text{finite volume}}\right],$$

$$T_{\ell r} = \frac{\beta_0 g^2}{16\pi^2}\left[\delta_{\ell r}\ln\left(\frac{e^{5/3}\mu^2}{p^2}\right) + \underbrace{f_0(p^2R^2)_{\ell r}}_{\text{finite volume}}\right]$$

  - Large-β_0 approximation [Beneke–Braun '94]: Radiative corrections of A_μ are included by $-\frac{2}{3}n_W \to \beta_0 = (\frac{11}{3} - \frac{2}{3}n_W)N$

- If $\Pi(p^2 \to 0) \sim \ln p^2$, there exists the renormalon.

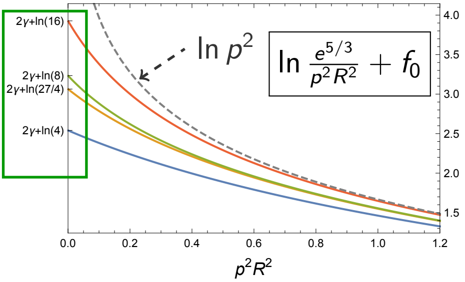
Absence of renormalons on $\mathbb{R}^3 \times S^1$

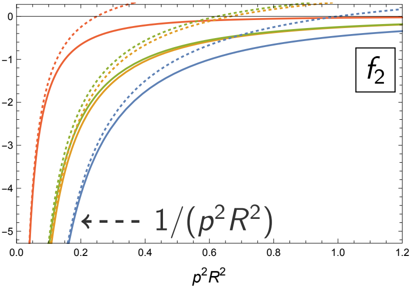
- Asymptotic behavior of L and T in $SU(^\forall N)$ QCD(adj.)

$$L_{\ell r}(p^2R^2) = -\frac{m_{\mathrm{sc}}^2}{p^2}\delta_{\ell r} + \text{const.} + \mathcal{O}(p^2R^2\ln(p^2R^2)),$$

$$T_{\ell r}(p^2R^2) = \text{const.} + \mathcal{O}(p^2R^2\ln(p^2R^2)), \qquad m_{\mathrm{sc}}^2 \propto \frac{\beta_0 g^2}{16\pi^2 R^2}$$

  - $N = 2\&3$: identical to [Anber-Sulejmanpasic] ($\beta_0 \leftrightarrow \frac{2}{3}(1-n_W)$)





- No logarithmic factor → no renormalons (no factorial growth)

Subtlety of large N limit on $\mathbb{R}^3 \times S^1$

- Subtlety of naive (apparent) $1/N$ expansion: e.g.,

$$|f_0| \lesssim \frac{1}{N}\frac{1}{(p^2R^2)^{3/2}} \quad \Rightarrow \quad \ln p^2 + f_0(p^2) \overset{\text{finite } p}{\sim} \ln p^2 + \mathcal{O}(1/N)$$

$\Pi(p^2) \overset{N\to\infty}{\to} \ln p^2$; $^\exists$renormalons [Ashie-O.M.-Suzuki-Takaura-Takeuchi]

- Does there exist renormalons in "$N \to \infty$"?

$\boxed{N \ll \text{IR divergence}}$

$$
\begin{tikzcd}
\Pi(p^2) \arrow[rrr, "p\to 0"] \arrow[d, "N\to\infty"'] & & & \text{const.} \arrow[dr] & \\
 & & & & \boxed{\nexists \text{ for } \forall N} \\
\ln p^2 \arrow[r, "p\to 0"'] & \boxed{\exists} \arrow[rr, "p_d\sim\frac{1}{NR}"] \arrow[rrd, "p_d\sim\frac{1}{R}"'] & & \boxed{\Lambda^\alpha \text{ (volume independence)}} & \\
 & & & \boxed{\Lambda^{\alpha-1}/R \text{ [Ishikawa \textit{et. al.}]}} &
\end{tikzcd}
$$

$\boxed{N \gg \text{IR divergence}}$

- Volume independence: "$N \gg \infty$" and twisted loop momentum
- Definition of "renormalon" $\overset{\text{Incompatible?}}{\longleftrightarrow}$ $1/N$ expansion

Backup: Twisted BC, volume (in)dependence

- Identity for KK sum with twisted BC:

$$\frac{1}{2\pi R}\sum_{p_d\in\mathbb{Z}/R} F(\boldsymbol{p}, p_d + m_A) = \sum_{n\in\mathbb{Z}}\int\frac{dp_d}{2\pi}\underbrace{e^{ip_d 2\pi Rn}} F(\boldsymbol{p}, p_d + m_A)$$

$$= \sum_{n\in\mathbb{Z}}\int\frac{dp_d}{2\pi}\underbrace{e^{i(p_d - m_A)2\pi Rn}} F(\boldsymbol{p}, p_d)$$

- Noting that

$$\sum_A e^{2\pi i m_A Rn}\times 1 = \begin{cases} N, & \text{for } n = 0 \bmod N \text{ (leading)},\\ 0, & \text{for } n \neq 0 \bmod N,\end{cases}$$

the effective radius becomes NR:

$$\xrightarrow{\text{sum over } A} N\sum_{n'\in\mathbb{Z}}\int\frac{dp_d}{2\pi}\underbrace{e^{ip_d 2\pi NRn'}} F(\boldsymbol{p}, p_d) = \frac{N}{2\pi R}\sum_{p_d\in\mathbb{Z}/(NR)} F(\boldsymbol{p}, p_d)$$

- Assume "large N limit" as $NR\Lambda\to\infty \Rightarrow$ Decompactification
- Volume *dependence*: $\underbrace{\text{other } m_A \text{ dependence}}_{E \text{ in SUSY } \mathbb{C}P^N}$, $\underbrace{\cancel{\text{twisted}}\text{ loop momenta}}_{e.g., U(1) \text{ gauge field}}$

Renormalon from momentum integration on $\mathbb{R}^4$

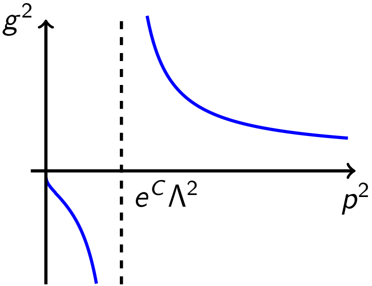
- $\delta_{\text{renormalon}}\big|_{\mathbb{R}^3\times S^1} = 0$ for $\forall R$, but $\delta_{\text{renormalon}}\big|_{\mathbb{R}^4} = e^{-16\pi^2/(\beta_0 g^2)}$
- How does renormalon on $\mathbb{R}^4$ emerge under $R \to \infty$?
- Reconsider renormalon diagram; resumming the geometric series,

$$g^2(\mu^2)\int \frac{d^4p}{(2\pi)^4}\frac{(p^2)^\alpha}{1-\Pi(p^2)}$$

$$= \int \frac{d^4p}{(2\pi)^4}\frac{(p^2)^\alpha}{\frac{\beta_0}{16\pi^2}\ln\frac{p^2}{e^C\Lambda^2}} = \int \frac{d^4p}{(2\pi)^4}(p^2)^\alpha g^2(p^2 e^{-C})$$

- Pole singularity at $p^2 = e^C\Lambda^2$
  $\to$ Contour deformation in $p^2 \in \mathbb{C}$
  $\to$ Renormalon $\sim \pm i\pi\frac{1}{\beta_0}(e^C\Lambda^2)^{\alpha+2}$

  - On $\mathbb{R}^4$, ambiguity of $\int d^4p$
    $=$ ambiguity in Borel sum



"Renormalon precursor" and decompactification

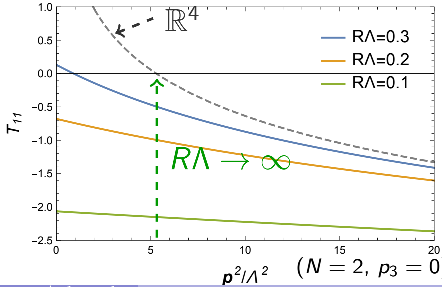
- In $SU(N)$ QCD(adj.) on $\mathbb{R}^3 \times S^1$,

$$g^2(\mu^2) \int \frac{d^3p}{(2\pi)^4} \frac{1}{2\pi R} \sum_{p_3 \in \mathbb{Z}/R} \frac{(p^2)^\alpha}{1 - \Pi(p^2)}$$

$$= \int \frac{d^3p}{(2\pi)^4} \frac{1}{2\pi R} \sum_{p_3 \in \mathbb{Z}/R} \frac{\frac{16\pi^2}{\beta_0}(p^2)^\alpha}{\ln \frac{p^2}{e^{5/3}\Lambda^2} + f_{\text{finite}}}$$

- "Renormalon precursor" comes from $\ln \frac{p^2}{e^{5/3}\Lambda^2} + f_{\text{finite}} = 0$
  - For $p_3 \lesssim \Lambda$, it can exist (no Borel singularities)
  - $\boxed{R \to \infty}$ $f_{\text{finite}} \to 0$ $(\text{pole}|_{\mathbb{R}^3 \times S^1} \to \text{pole}|_{\mathbb{R}^4})$ and $\sum_{p_3} \to \int dp_3$

    Renormalon on $\mathbb{R}^4$ emerge!



$(N = 2,\ p_3 = 0)$

Backup: toy model (large N volume independence)

- Difficult to compute the renormalon precursor directly
- A simplification by large N volume independence: For $|p_3| < e^{5/6}\Lambda$, integrand possesses a simple pole

$$\overset{p^2 \sim e^{5/3}\Lambda^2}{\to} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\pi R} \sum_{|p_3|<e^{5/6}\Lambda} \frac{16\pi^2}{\beta_0} (p^2)^\alpha \frac{e^{5/3}\Lambda^2}{p^2 - e^{5/3}\Lambda^2}$$

$$\sim \pm i\pi \frac{4}{\beta_0} \frac{(e^{5/3}\Lambda^2)^{\alpha+2}}{2\pi} \frac{1}{e^{5/3}R\Lambda} \sum_{|n|<e^{5/6}R\Lambda} \sqrt{1 - \frac{n^2}{(e^{5/3}R\Lambda)^2}}$$

- Decompactification $R\Lambda \to \infty$

$$\to \pm i\pi \frac{4}{\beta_0} \frac{(e^{5/3}\Lambda^2)^{\alpha+2}}{2\pi} \int_{-1}^{1} dx \sqrt{1 - x^2} = \text{Renormalon on } \mathbb{R}^4$$

- $[1 - \Pi(p \to 0)] \times [1 - \Pi(p \to \infty) < 0]$: $^\exists$Renormalon precursor

Contents

Summary

- (Our) Current understanding on resurgence structure in QFT

	PT	Semi-classical object
$\mathbb{R}^4$	$\delta_{\text{PFD}} \sim e^{-16\pi^2/g^2}$	$\delta_{\text{instanton}} \sim e^{-2S_I}$
$\mathbb{R}^4$	$\delta_{\text{renormalon}} \sim e^{-16\pi^2/(\beta_0 g^2)}$	?
$\mathbb{R}^3 \times S^1$	$\delta'_{\text{PFD}} \sim e^{-16\pi^2/(Ng^2)}$	$\delta_{\text{bion}} \sim e^{-2S_I/N}$
$\mathbb{R}^3 \times S^1$	$\delta_{\text{renormalon}} = 0$	No

- Renormalon precursor smoothly reduces renormalon in $R \to \infty$
  - Not associated with Borel singularities
  - Analytic continuation of geometric series
- There remains renormalon puzzle
  - What definition do you prefer? [talk by Cherman]
- Helpful in giving a unified understanding on resurgence

Backup: Borel resummation (notation)

- Borel resummation: summing divergent asymptotic series

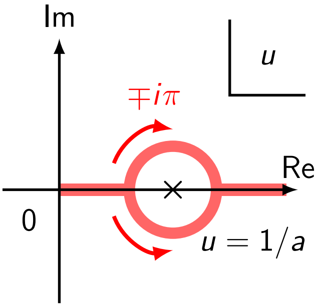
$$f(g^2) \sim \sum_{k=0}^{\infty} f_k \left(\frac{g^2}{16\pi^2} \right)^{k+1} \qquad \text{with } f_k \sim a^k k! \text{ as } k \to \infty$$

$\Downarrow$ Borel transform

$$B(u) \equiv \sum_{k=0}^{\infty} \frac{f_k}{k!} u^k = \frac{1}{1 - au} \qquad \text{(Pole singularity at } u = 1/a\text{)}.$$

- The Borel sum is given by

$$f(g^2) \equiv \int_0^{\infty} du\, B(u) e^{-16\pi^2 u/g^2}.$$



- $a < 0$ (alternating series) $\to$ convergent
- $a > 0 \to$ ill-defined due to the pole

  $\Rightarrow$ Imaginary ambiguity $\sim \pm e^{-16\pi^2/(ag^2)}$