On exact-WKB, resurgent structure, and the quantization conditions (arXiv:2008.00379 [hep-th])

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Introduction

There are two resurgence in physics.

1. P-NP relation (for transseries)

$$Z(\hbar) = \int_{periodic} \mathcal{D}x e^{-\frac{S[x]}{\hbar}} \tag{1}$$

$$= \sum_n a_n \hbar^n + e^{-\frac{S_1}{\hbar}} \sum_n b_n \hbar^n + e^{-\frac{S_2}{\hbar}} \sum_n c_n \hbar^n + \dots \tag{2}$$

The series is not converge, but asymptotic. The Borel ambiguity derived from the perturbative series has nonperturbative information.

$$(\mathcal{S}_+ - \mathcal{S}_-) \left[\sum_n a_n \hbar^n \right] \propto \pm i e^{-\frac{S_1}{\hbar}} \tag{3}$$

2. Exact WKB method (for differential equation)

$$\left(-\frac{\hbar^2}{2}\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\psi(x)\,. \tag{4}$$

$$\psi(x) = \sum_n \psi_n(x)\hbar^n \tag{5}$$

Then $\psi(x)$ is asymptotic series. If we consider its Borel summation and its analytic continuation,

$$\psi_{\mathrm{I}}^{+}(x) \to \psi_{\mathrm{II}}^{+}(x) + \psi_{\mathrm{II}}^{-}(x) \tag{6}$$

Riemann-Hilbert problem of the differential equation.

How are they related each other?

Other fundamental problems:

1. The relation among several quantization methods:

   - Bohr-Sommerfeld quantization
   - Schrödinger eq.
   - path integral
   - Gutzwiller trace formula
     e.g. Can we derive path integral from Bohr-Sommerfeld?

2. How to determine the intersection number of Lefschetz thimble (relevant saddle points)

The all questions are solved.

Contents

Exact WKB

Exact WKB

In WKB analysis, we consider the ansatz given by

$$\left(-\frac{\hbar^2}{2}\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E\psi(x)\,. \tag{7}$$

$$\psi(x,\hbar) = e^{\int^x S(x,\eta)dx}\,, \tag{8}$$

$$S(x,\hbar) = \hbar^{-1}S_{-1}(x) + S_0(x) + \hbar S_1(x) + \hbar^2 S_2(x) + \ldots \tag{9}$$

$$= S_{odd} + S_{even} \tag{10}$$

Then Schrödinger eq. becomes Riccati eq.

$$S(x)^2 + \frac{\partial S}{\partial x} = \hbar^{-2}Q(x)\,, \tag{11}$$

where $Q(x) = S_{-1} = \sqrt{2(V(x) - E)}$. Also we can show

$$S_{even} = -\frac{1}{2}\frac{\partial}{\partial x}\log S_{odd}\,. \tag{12}$$

Therefore the WKB wave function is expressed as

$$\psi_a^{\pm}(x) = e^{\int^x S^{\pm} dx} = \frac{1}{\sqrt{S_{odd}}} e^{\pm \int_a^x S_{odd} dx} \tag{13}$$

At the leading order, this expression becomes usual WKB approximation:

$$\psi_a^{\pm}(x) \sim \frac{1}{Q(x)^{1/4}} e^{\pm \frac{1}{\hbar} \int_a^x \sqrt{Q(x)} dx}\,, \tag{14}$$

Now, we take Borel summation of $\psi_a^{\pm}(x)$.

The posistion of Borel singularity depends on x.
$(\psi(x) = \sum a_n(x)\hbar^n)$

$\rightarrow$ *Stokes curve* tells where the Stokes phenomena happens.

Stokes curve is defined as

$$\mathrm{Im}\,\frac{1}{\hbar}\int_a^x \sqrt{Q(x)}dx = 0 \tag{15}$$

$$\tag{16}$$

$(Q(a) = 2(V(x) - E)\big|_{x=a} = 0$ i.e. turning point)

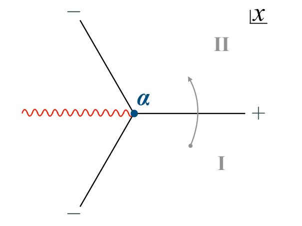


Figure 1: Airy: $V(x) = x$, across anti-clockwisely

$$\psi^{+}_{a,\mathrm{I}} = \psi^{+}_{a,\mathrm{II}} + i\psi^{-}_{a,\mathrm{II}}$$
$$\psi^{-}_{a,\mathrm{I}} = \psi^{-}_{a,\mathrm{II}}$$

When a wavefunction crosses a Stokes curve, its Stokes phenomena can be expressed as

$$\begin{pmatrix} \psi_{a,\mathrm{I}}^{+} \\ \psi_{a,\mathrm{I}}^{-} \end{pmatrix} = M \begin{pmatrix} \psi_{a,\mathrm{II}}^{+} \\ \psi_{a,\mathrm{II}}^{-} \end{pmatrix}, \tag{17}$$

where the the matrix M is given by

$$M = \begin{cases} \begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix} =: M_{+} & \text{for anti-clockwisely, } + \\ \begin{pmatrix} 1 & -i \\ 0 & 1 \end{pmatrix} =: M_{+}^{-1} & \text{for clockwisely, } + \\ \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix} =: M_{-} & \text{for anti-clockwisely, } - \\ \begin{pmatrix} 1 & 0 \\ -i & 1 \end{pmatrix} =: M_{-}^{-1} & \text{for clockwisely, } - \end{cases} \tag{18}$$

If one considers two wave functions normalized at the different turning points: a_1, a_2. They are related by

$$\psi^{\pm}_{a_1}(x) = e^{\pm \int_{a_1}^{a_2} S_{odd}} \psi^{\pm}_{a_2}(x) \tag{19}$$

Therefore

$$\begin{pmatrix} \psi^{+}_{a_1}(x) \\ \psi^{-}_{a_1}(x) \end{pmatrix} = N_{a_1 a_2} \begin{pmatrix} \psi^{+}_{a_2}(x) \\ \psi^{-}_{a_2}(x) \end{pmatrix}, \qquad N_{a_1 a_2} = \begin{pmatrix} e^{+\int_{a_1}^{a_2} S_{odd}} & 0 \\ 0 & e^{-\int_{a_1}^{a_2} S_{odd}} \end{pmatrix}. \tag{20}$$

N is called Voros multiplier. Actually we can derive the eigenvalues and also the partition function with these tools without solving Schrödinger eq.

Example: Harmonic Oscillator

Let the potential as $V(x) = \frac{1}{2}\omega^2 x^2$. Then its Stokes curve looks as follows ($E > 0$):

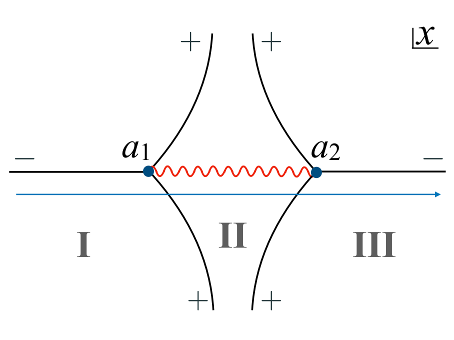


Figure 2:

where $a_1 = -\frac{\sqrt{2E}}{\omega}$, $a_2 = \frac{\sqrt{2E}}{\omega}$ are turning points. The blue line is the path of analytic continuation.

First,

$$\begin{pmatrix} \psi^{+}_{a_1,\mathrm{I}}(x) \\ \psi^{-}_{a_1,\mathrm{I}}(x) \end{pmatrix} = M_+ \begin{pmatrix} \psi^{+}_{a_1,\mathrm{II}}(x) \\ \psi^{-}_{a_1,\mathrm{II}}(x) \end{pmatrix} \tag{21}$$

Second,

$$\begin{pmatrix} \psi^{+}_{a_1,\mathrm{II}}(x) \\ \psi^{-}_{a_1,\mathrm{II}}(x) \end{pmatrix} = N_{a_1 a_2} \begin{pmatrix} \psi^{+}_{a_2,\mathrm{II}}(x) \\ \psi^{-}_{a_2,\mathrm{II}}(x) \end{pmatrix} \tag{22}$$

Then

$$\begin{pmatrix} \psi^{+}_{a_2,\mathrm{II}}(x) \\ \psi^{-}_{a_2,\mathrm{II}}(x) \end{pmatrix} = M_+ \begin{pmatrix} \psi^{+}_{a_2,\mathrm{III}}(x) \\ \psi^{-}_{a_2,\mathrm{III}}(x) \end{pmatrix} \tag{23}$$

After all ($A = e^{\oint_A S_{odd}} = e^{2\int_{a_1}^{a_2} S_{odd}}$)

$$\begin{pmatrix} \psi^{+}_{a_1,\mathrm{I}}(x) \\ \psi^{-}_{a_1,\mathrm{I}}(x) \end{pmatrix} = M_+ N_{a_1 a_2} M_+ N_{a_2 a_1} \begin{pmatrix} \psi^{+}_{a_1,\mathrm{III}}(x) \\ \psi^{-}_{a_1,\mathrm{III}}(x) \end{pmatrix} \tag{24}$$

$$= \begin{pmatrix} \psi^{+}_{a_1,\mathrm{III}}(x) + i(1+A)\psi^{-}_{a_1,\mathrm{III}}(x) \\ \psi^{-}_{a_1,\mathrm{III}}(x) \end{pmatrix} \tag{25}$$

therefore we obtain

$$D = 1 + A = 0 \tag{26}$$

This is equivalent to

$$\oint_A S_{odd} = n + \frac{1}{2} \quad \text{with } n \in \mathbb{Z} \tag{27}$$

In the case of harmonic oscillator, we can show

$$\oint_A S_{odd} = \frac{1}{\hbar} \oint_A \sqrt{2(V(x) - E)}dx = -2\pi i \frac{E}{\hbar\omega} \tag{28}$$

(The higher orders of S_{odd} don't contribute to this integral)
Therefore $D(E) = 0$ gives

$$E = \hbar\omega \left(n + \frac{1}{2} \right) \tag{29}$$

$$\tag{30}$$

the condition of Stokes curve: $E > 0$ determines $n = 0, 1, 2...$

Resolvent method

- bridge exact WKB to the partition function and Gutzwiller

Resolvent method

We can regard the quantization condition derived from exact WKB: $D(E) = 0$ as the Fredholm determinant: $D = \det\left(\hat{H} - E\right)$.

For the trace of resolvent: $G(E) = \operatorname{tr} \frac{1}{H-E}$, it can be expressed as $-\frac{\partial}{\partial E} \log D = G(E)$. Also

$$G(E) = \int_0^\infty Z(\beta) e^{\beta E} d\beta \tag{31}$$

$$Z(\beta) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} G(E) e^{-\beta E} dE \,, \tag{32}$$

where $Z(\beta) = \operatorname{tr} e^{-\beta H}$

Indeed,

$$D = 1 + A = 1 + e^{-2\pi i \frac{E}{\hbar\omega}} \tag{33}$$

$$= e^{-\pi i \frac{E}{\hbar\omega}} 2 \sin\left(\pi\left(\frac{E}{\hbar\omega} + \frac{1}{2}\right)\right) \tag{34}$$

$$= e^{-\pi i \frac{E}{\hbar\omega}} \frac{2\pi}{\Gamma(\frac{1}{2} + \frac{E}{\hbar\omega})\Gamma(\frac{1}{2} - \frac{E}{\hbar\omega})} \tag{35}$$

$$G(E) = -\frac{\partial}{\partial E} \log(1 + A) \tag{36}$$

$$= -\frac{\partial}{\partial E} \log\left(e^{-\pi i \frac{E}{\hbar\omega}} \frac{2\pi}{\Gamma(\frac{1}{2} + \frac{E}{\hbar\omega})\Gamma(\frac{1}{2} - \frac{E}{\hbar\omega})}\right) \tag{37}$$

The partition function is

$$Z = \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} -\frac{\partial}{\partial E}\log\left(e^{-\pi i \frac{E}{\hbar\omega}}\frac{2\pi}{\Gamma(\frac{1}{2}+\frac{E}{\hbar\omega})\Gamma(\frac{1}{2}-\frac{E}{\hbar\omega})}\right)e^{-\beta E}dE \tag{38}$$

$$= \sum_{n=0}^{\infty} e^{-\beta\hbar\omega\left(n+\frac{1}{2}\right)} \tag{39}$$

Remark: We don't have to solve the Schrödinger eq. or path integral to derive the partition function.

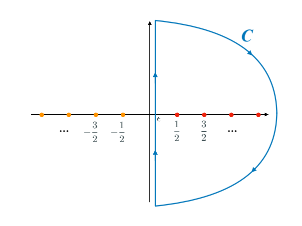


Figure 3: C is the integration contour

Also,

$$G(E) = -\frac{\partial}{\partial E}\log(1+A) = \frac{-\frac{\partial}{\partial E}A}{1+A} \tag{40}$$

$$= \frac{i}{\hbar}\sum_{n=1}^{\infty} T e^{\frac{in}{\hbar}\oint_A p dx}(-1)^n, \tag{41}$$

(where T is the period of harmonic oscillator)

This is actually the *Gutzwiller trace formula* of harmonic oscillator.

Gutzwiller trace formula

Gutzwiller trace formula

$$Z(T) = \operatorname{tr} e^{-iHT} \tag{42}$$
$$= \int_{periodic} \mathcal{D}x \; e^{iS} \tag{43}$$

$$G(E) = \int_0^\infty Z(T) e^{(iE-\epsilon)T} dT = -i \operatorname{tr} \frac{1}{H-E} \tag{44}$$

therefore

$$G(E) = -i \operatorname{tr} \frac{1}{H-E} = \int_0^\infty dT \int_{periodic} \mathcal{D}x \; e^{iS+iET} \tag{45}$$
$$= \int_0^\infty dT \int_{periodic} \mathcal{D}x \; e^{i\Gamma}, \tag{46}$$

where $\Gamma = S + ET$. Action, S can be written as

$$S = \int p\dot{x}dt - \int^{T} H dt \tag{47}$$

$$= \oint pdx - \int^{T} H dt \tag{48}$$

Evaluate T integral by stationary phase method

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}T} = \frac{\mathrm{d}S}{\mathrm{d}T} + E \tag{49}$$

Using $\frac{\mathrm{d}}{\mathrm{d}T} \oint pdx = 0$,

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}T} = \frac{\mathrm{d}S}{\mathrm{d}T} + E = -H + E \tag{50}$$

The leading contributions are periodic classical solutions whose energy is E. There are n-times periodic orbit too: $\oint pdx \to n \oint pdx$.

$$\Gamma = S + ET = \left(n \oint pdx - ET \right) + ET = n \oint pdx \quad (n = 1, 2, 3...) \tag{51}$$

Finally,

$$G(E) = \sum_{p.p.o.} \sum_{n=1}^{\infty} e^{in \oint_{p.p.o.} pdx} \tag{52}$$

$p.p.o.$ stands for *prime periodic orbit*, which is a topologically distinguishable orbit among the countless periodic orbits.

If we consider sub-leading term of stationary phase approximation,

$$G(E) \simeq i \sum_{p.p.o.} \sum_{n=1}^{\infty} T(E) e^{in \oint_{p.p.o.} pdx} (-1)^n \left(\det \left| \frac{\delta^2 S}{\delta x_i \delta x_j} \right| \right)^{-1/2} \tag{53}$$

where $i(-1)^n$ is the *Maslov index*.

Maslov index

Maslov index is the index determined by the number of negative eigenvalue of M, where

$$M = \left.\frac{\delta^2 S}{\delta x \delta x}\right|_{x=x_{cl}} = -\frac{\mathrm{d}^2}{\mathrm{d}t^2} - V''(x_{cl})\,. \tag{54}$$

$$\sqrt{\det M} = \sqrt{|\det M|}e^{i\alpha\pi}, \qquad \alpha = \frac{\nu}{2}\,. \tag{55}$$

Here, α is called the *Maslov index*. (ν is the number of negative eigenvalues of M) The determinant of the n-cycle is given by

$$\sqrt{\det M} = -i\sqrt{|\det M|}(-1)^n\,. \tag{56}$$

Because the operator M has $2n-1$ negative eigenvalues for n-cycle orbit. (and we call this $(-1)^n$ as Maslov index from here)

Proof

Consider classical EoM:

$$-\frac{\mathrm{d}^2 x_{cl}}{\mathrm{d}t^2} - \frac{\mathrm{d}V}{\mathrm{d}x_{cl}} = 0\,. \tag{57}$$

Take t differential for this equation. Then we get

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}t^2} - V''(x_{cl})\right)\frac{\mathrm{d}x_{cl}}{\mathrm{d}t} = 0\,. \tag{58}$$

This expression is nothing but an eigenvalue equation for the zero eigenvalue of the fluctuation operator, $M\tilde{\psi}_0(t) = 0$, and the eigenfunction is proportional to $\tilde{\psi}_0(t) = \frac{\mathrm{d}x_{cl}}{\mathrm{d}t}$.

Next, let us consider a periodic classical solution x_{cl}. When it is a one-cycle solution, the derivative $\frac{\mathrm{d}x_{cl}}{\mathrm{d}t}$ typically has a behavior depicted in Fig. 4.

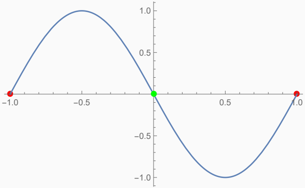


Figure 4: The appearance of the derivative $\frac{\mathrm{d}x_{cl}}{\mathrm{d}t}$ for 1-cycle.

The operator M is a Schrödinger-type operator, thus this is the first excited state so there is one negative eigenvalue.

Similary, 2-cycle is the third excited state so there are three negative eigenvalues...
$\rightarrow M$ has $2n-1$ negative eigenvalues for n-cycle orbit.

Harmonic oscillator in Gutzwiller's form

There is only one type of p.p.o. with constant $T(E)$ and $|\det \frac{\delta^2 S}{\delta x_i \delta x_j}|$, We then obtain

$$G(E) \propto \sum_{n=1}^{\infty} e^{in \oint pdx} (-1)^n = \frac{e^{i \oint pdx}}{1 + e^{i \oint pdx}} \,. \tag{59}$$

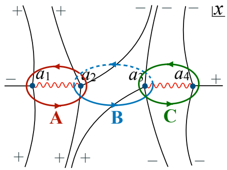
This is same to the $G(E)$ obtained from exact WKB, and the poles of $G(E)$ are given by

$$\oint pdx = 2\pi \left(n + \frac{1}{2} \right) . \tag{60}$$

However, the way to determine p.p.o. and how to sum them up were not known in general cases.

$\rightarrow$ As we will show, you can identify them exactly!

Double well potential



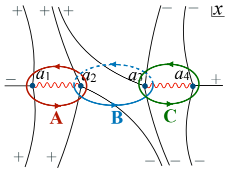


Figure 5: left: $\operatorname{Im} \hbar > 0$, right: $\operatorname{Im} \hbar < 0$

When $\operatorname{Im} \hbar > 0$

$$D^{+} = (1 + A)(1 + C) + AB = 0\,. \tag{61}$$

When $\operatorname{Im} \hbar < 0$

$$D^{-} = (1 + A)(1 + C) + CB = 0\,. \tag{62}$$

$A = e^{\oint_A S_{odd}}$, $B = e^{\oint_B S_{odd}}, C = e^{\oint_C S_{odd}} = A^{-1}$. $B \propto e^{-\frac{S}{\hbar}}$

DDP(Delabaere-Dillinger-Pham) formula

In our case, (Please see our paper for the generic case)

$$\mathcal{S}_+[A] = \mathcal{S}_-[A](1+\mathcal{S}[B])^{-1}, \tag{63}$$

$$\mathcal{S}_+[B] = \mathcal{S}_-[B] =: \mathcal{S}[B], \tag{64}$$

$$\mathcal{S}_+[C] = \mathcal{S}_-[C](1+\mathcal{S}[B])^{+1}, \tag{65}$$

Using this formula, we can show

$$\mathcal{S}_+[D^+] = \mathcal{S}_-[D^-] \tag{66}$$

i.e. both are equivalent when we take Borel summation.

Analyse the exact quantization condition

let us consider the asymptotic form of A

$$A \to e^{-2\pi i \frac{E}{\hbar \omega_A(E,\hbar)}} . \tag{67}$$

This $\omega_A(E,\hbar)$ is an asymptotic expansion in $\hbar$.

$$\omega_A(E,\hbar)^2 = \sum_{n=0}^{\infty} c_n(E)\hbar^n \tag{68}$$

$$\lim_{E\to 0} c_0(E) = V''(x_{\rm vac}) , \tag{69}$$

Then $D^{\pm}$ becomes $(E = \hbar\omega_A(\frac{1}{2}+\delta))$

$$\begin{aligned} 4\sin^2(\pi\delta) &= e^{-2\pi i\delta}B \qquad \operatorname{Im}\hbar > 0 , \\ 4\sin^2(\pi\delta) &= e^{2\pi i\delta}B \qquad \operatorname{Im}\hbar < 0 . \end{aligned} \tag{70}$$

Or equivalently,

$$\begin{aligned} \frac{1}{\Gamma(-\delta)} &= \pm\frac{\sqrt{B}}{2\pi}e^{-\pi i\delta}\Gamma(1+\delta) \qquad \operatorname{Im}\hbar > 0 , \\ \frac{1}{\Gamma(-\delta)} &= \pm\frac{\sqrt{B}}{2\pi}e^{\pi i\delta}\Gamma(1+\delta) \qquad \operatorname{Im}\hbar < 0 . \end{aligned} \tag{71}$$

ZINN-Justin's result from the path integral is

$$\frac{1}{\Gamma(-x)} = \pm\frac{e^{-S_{inst}}}{2\pi}e^{-\pi i x}\left(\frac{\hbar}{2}\right)^{-x-\frac{1}{2}}\sqrt{2\pi} \qquad \mathrm{Im}\,\hbar < 0\,.$$
$$\frac{1}{\Gamma(-x)} = \pm\frac{e^{-S_{inst}}}{2\pi}e^{\pi i x}\left(\frac{\hbar}{2}\right)^{-x-\frac{1}{2}}\sqrt{2\pi} \qquad \mathrm{Im}\,\hbar < 0\,., \tag{72}$$

where $x = E - \frac{1}{2}$. Considering that $\left(\frac{\hbar}{2}\right)^{-\delta-\frac{1}{2}}\sqrt{2\pi}$ in (72) is the contribution from quantum fluctuations, this part is included in B and ω_A in (71). The extra Gamma function $\Gamma(1+\delta)$ is, essentially coming from the negative energy part.

(Note: Using degenerate Weber-type exact WKB, we can produce this result completely)

$(D^{+}\colon \operatorname{Im}\hbar > 0,\ D^{-}\colon \operatorname{Im}\hbar < 0)$
$D^{\pm} = (1+A)(1+C) + AB = (1+A)(1+A^{-1})(1+\frac{B}{(1+A^{\mp})^2})$
Using $G(E) = -\frac{\partial}{\partial E}\log D$,

$$G(E) = G_p(E) + G_{np}(E) \tag{73}$$

$$\tag{74}$$

$$G_p(E) = -\frac{\partial}{\partial E}\log(1+A) - \frac{\partial}{\partial E}\log\left(1+\frac{1}{A}\right) \tag{75}$$

$$G_{np}(E) = -\frac{\partial}{\partial E}\log\left(1+\frac{B}{(1+A^{\mp})^2}\right) \tag{76}$$

$$= -\frac{\partial}{\partial E}\log\left(1+\frac{B}{(D_A^{\pm})^2}\right) \tag{77}$$

The derivative term $\frac{\partial}{\partial E} A$ produces the "period"

$$\frac{\partial}{\partial E} A = \frac{\partial}{\partial E} e^{\oint_A S_{\mathrm{odd}}} = \left(\frac{\partial}{\partial E} \oint_A S_{\mathrm{odd}} \right) e^{\oint_A S_{\mathrm{odd}}}$$
$$= \left(\oint_A \frac{1}{\hbar} \frac{-1}{\sqrt{2(V-E)}} + O(\hbar) \right) e^{\oint_A S_{\mathrm{odd}}} \equiv -\frac{1}{\hbar} i T_A A \,. \quad (78)$$

and similarly,

$$\frac{\partial}{\partial E} B = -\frac{1}{\hbar} i T_B B \,. \quad (79)$$

T_A is real, and T_B is pure imaginary.

Using these quantities, $G(E)$ can be expressed as

$$G(E) = G_p + G_{np} \tag{80}$$

$$G_p(E) = i\frac{1}{\hbar}T_A\sum_{n=1}^{\infty}(-1)^n A^n + i\frac{1}{\hbar}T_A\sum_{n=1}^{\infty}(-1)^n A^{-n}, \tag{81}$$

$$G_{np}(E) = -\frac{\partial}{\partial E}\big((D_A^{\pm})^{-2}B\big)\sum_{n=1}^{\infty}(-1)^n((D_A^{\pm})^{-2}B)^n, \tag{82}$$

$$(D^{\pm})_A^{-2}B = \begin{cases} B\big(\sum_{k=1}^{\infty}(-1)^k A^{-k}\big)\big(\sum_{l=1}^{\infty}(-1)^l A^{-l}\big) & (\mathrm{Im}\,\hbar > 0) \\ B\big(\sum_{k=1}^{\infty}(-1)^k A^{k}\big)\big(\sum_{l=1}^{\infty}(-1)^l A^{l}\big) & (\mathrm{Im}\,\hbar < 0) \end{cases} \tag{83}$$

$$\frac{\partial}{\partial E}\big((D_A^{\pm})^{-2}B\big) = -i\frac{1}{\hbar}\sum_{n,m=1}^{\infty}(-1)^{(n+m)}\big(T_B \mp (n+m)T_A\big)B(A^{\mp})^{n+m}, \tag{84}$$

This is exactly the form of Gutzwiller trace formula and the factor $(-1)^n$ is regarded as the Maslov index.

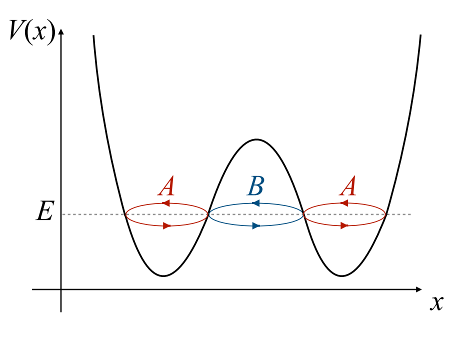


Figure 6:

$$G_{np} \sim \sum_{n=1}^{\infty} (-1)^n (D_A^{-2} B)^n \tag{85}$$

$$D_A^{-2} B = \begin{cases} B\left(\sum_{k=1}^{\infty} (-1)^k A^{-k}\right)\left(\sum_{l=1}^{\infty} (-1)^l A^{-l}\right) & (\mathrm{Im}\,\hbar > 0) \\ B\left(\sum_{k=1}^{\infty} (-1)^k A^{k}\right)\left(\sum_{l=1}^{\infty} (-1)^l A^{l}\right) & (\mathrm{Im}\,\hbar < 0) \end{cases} \tag{86}$$

Partition function and QMI

$$G(E) = G_p(E) + G_{np}(E) \tag{87}$$

$$\tag{88}$$

$$G_p(E) = -\frac{\partial}{\partial E}\log(1+A) - \frac{\partial}{\partial E}\log\left(1+\frac{1}{A}\right) \tag{89}$$

$$G_{np}(E) = -\frac{\partial}{\partial E}\log\left(1+\frac{B}{(D_A^{\pm})^2}\right) \tag{90}$$

$$Z(\beta) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty} G(E)e^{-\beta E}dE \tag{91}$$

$$Z = Z_p(\beta) + Z_{np}(\beta) \tag{92}$$

$$Z_{np}(\beta) = \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\left[-\frac{\partial}{\partial E}\log\left(1+\frac{B}{(D_A^{\pm})^2}\right)\right]e^{-\beta E}dE \tag{93}$$

$$= -\beta\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\log\left(1+\frac{B}{(D_A^{\pm})^2}\right)e^{-\beta E}dE \tag{94}$$

$$= \beta\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{B}{(D_A^{\pm})^2}\right)^n(-1)^n e^{-\beta E}dE \tag{95}$$

$B \propto e^{-\frac{S}{\hbar}}$, so this summation is indeed multi-bion contribution. Using DDP formula, we can say

$$\mathcal{S}_+[Z^+] = \mathcal{S}_-[Z^-] \tag{96}$$

i.e. we can identify the exact form of resurgent structure of the partition function!

QMI(quasi-moduli integral) form

Using $A \to e^{-2\pi i \frac{E}{\hbar\omega_A(E,\hbar)}}$, By defining $s \equiv E/(\hbar\omega_A) - 1/2$,

$$Z_{\rm np}(\beta) = \tag{97}$$

$$\beta \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} \sum_{n=1}^{\infty} \frac{1}{n}\left(B\Gamma(-s)^2 \frac{1}{2\pi} e^{\mp 2\pi i s}\right)^n e^{-\beta\frac{\hbar\omega_A}{2}} \hbar\omega_A e^{-s\beta} ds\,. \tag{98}$$

Here, the partition function obtained by calculating the path integral is as follows:

$$\frac{Z_{\rm np}}{Z_0} = \tag{99}$$

$$\beta \frac{1}{2\pi i}\int_{\epsilon - i\infty}^{\epsilon + i\infty} \sum_{n=1}^{\infty} \frac{1}{n}\left(e^{-S_{\rm bion}}\left(\frac{\det M_I}{\det M_0}\right)^{-1} \frac{S_{\rm inst}}{2\pi}\Gamma(-s)^2 \left(\frac{\hbar}{2}\right)^{-2s} e^{\mp 2\pi i s}\right)^n e^{-s\beta} ds\,. \tag{100}$$

$$QMI^n = \frac{1}{n}\prod_{i=1}^{2n}\left(\int_0^\infty d\tau_i e^{-\mathcal{V}_i(\tau_i)}\right)\delta\left(\sum_{k=1}^{2n}\tau_k - \beta\right)$$
$$= \frac{1}{2\pi i n}\int_{-i\infty}^{i\infty} ds e^{-s\beta}\left(e^{\pm i\pi(-s)}\left(\frac{\hbar}{2}\right)^{-s}\Gamma(-s)\right)^{2n} \quad (101)$$

From the path integral,

$$D(E) = \frac{1}{\Gamma(\frac{1}{2}-E)\Gamma(\frac{1}{2}-E)}\left(1 - Be^{\pm i\pi(1-2E)}\left(\frac{\hbar}{2}\right)^{(1-2E)}\Gamma\left(\frac{1}{2}-E\right)\Gamma\left(\frac{1}{2}-E\right)\right) \quad (102)$$

The first $\frac{1}{\Gamma(\frac{1}{2}-E)\Gamma(\frac{1}{2}-E)}$ are from two vacua and the latter ones are from QMI. This miracle is easily explained by this Gutzwiller's representation. Essentially both have the same origin, the infinite number of A cycles, $D_A^{-1} = \frac{1}{1+A} = \sum_n^\infty (-1)^n A^n \sim \Gamma(\frac{1}{2}-E)$.

The intersection number of Lefschetz thimble

The intersection number of Lefschetz thimble

First, we write down the partition function formally as a sum over saddle points

$$
\begin{aligned}
Z(\beta) &= \operatorname{tr} e^{-\beta \hat{H}} = \int \mathcal{D}x \; e^{-\frac{S[x]}{\hbar}} \\
&= n_0 \, \mathcal{S}\left[e^{-\frac{S[x_0]}{\hbar}} \sum_n a_n \hbar^n \right] + n_1 \, \mathcal{S}\left[e^{-\frac{S[x_1]}{\hbar}} \sum_n b_n \hbar^n \right] + \ldots \\
&= \sum_\sigma n_\sigma \int_{\mathcal{J}_\sigma} \mathcal{D}x \; e^{-\frac{S[x]}{\hbar}} = \sum_\sigma n_\sigma \; Z_\sigma(\beta) \,, \qquad (103)
\end{aligned}
$$

where $\mathcal{S}[\cdot]$ denotes the Borel summation of series expansions and x_σ stands for saddle points.

$$
\begin{aligned}
\operatorname{tr} \frac{1}{\hat{H}-E} = G(E) &= \int_0^\infty Z(\beta) e^{\beta E} \,\mathrm{d}\beta \\
&= \sum_\sigma n_\sigma \int_0^\infty Z_\sigma(\beta) e^{\beta E} \,\mathrm{d}\beta \\
&= \sum_\sigma n_\sigma G_\sigma(E)\,. \qquad (104)
\end{aligned}
$$

The trace of resolvent $G(E)$ can be connected to the Fredholm determinant $D(E) = \det\left(\hat{H} - E\right)$ via the relation $-\frac{\partial}{\partial E} \log D = G(E)$. Then, we have

$$
D(E) = \prod_\sigma D_\sigma^{n_\sigma}(E)\,, \qquad (105)
$$

where $D_\sigma(E)$ stands for the Fredholm determinant for each thimble.

Now, the quantization condition given by Eq. (61) can be rewritten as

$$D = (1+A)(1+A^{-1}) \prod_{n=1}^{\infty} D_n^{(-1)^n} \tag{106}$$

$$D_n = e^{-\frac{1}{n}\left(\frac{B}{D_A^2}\right)^n} \tag{107}$$

$$\begin{aligned} Z_n &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\left[\frac{\partial}{\partial E}\frac{1}{n}\left(\frac{B}{D_A^2}\right)^n\right] e^{-\beta E} dE \\ &= \frac{\beta}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\frac{1}{n}\left(\Gamma(-s)^2\frac{B}{2\pi}e^{\mp 2\pi i(1/2+s)}\right)^n e^{-\beta(\hbar\omega_A(1/2+s))}\hbar\omega_A ds\,. \end{aligned} \tag{108}$$

→ The Maslov index is regarded as the intersection number of Lefschetz thimble!

Summary

- We show how the two resurgence are related each other, and the Stokes phenomena of partition function (or energy) corresponds to the change of the “topology” of the Stoke curve.
- We show the cancellation of Borel ambiguity of partition function without approximation.
- We show the relation between the Maslov index and the intersection number of Lefschetz thimble and how to determine it.
- Using Exact WKB method, we show the exact relationship among Schrödinger eq., Bohr-Sommerfeld, Gutzwiller and path integral.
- (Generalizing to N-ple well potential, including higher genus systems.)

Discussion

- Exact WKB on S^1 with θ term $\rightarrow$ succeed! we can see the degenerate at $\theta = \pi$ too
- phase transition and complex turning point
- Degenerate Weber vs Airy-type exact WKB

$$D^{\pm} = (1+A_1)(1+A_2)(1+A_1^{-1}) \cdot \prod_{n=1}^{\infty} \exp\left[-\frac{1}{n}\left\{B\left(\frac{1}{D^{+}_{A_1}D^{\pm}_{A_2}} + \frac{1}{D^{-}_{A_1}D^{\pm}_{A_2}}\right) + \frac{B^2}{D^{+}_{A_1}D^{-}_{A_1}D^{\pm}_{A_2}}\right\}^n\right]^{(-1)^n}. \tag{109}$$

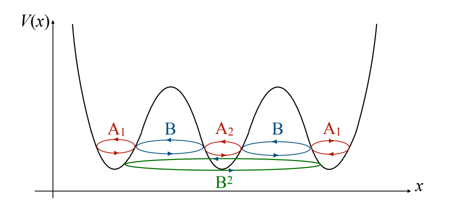


Figure 7:

$$Z = Z_{\rm p} + Z_{\rm np}\,, \tag{110}$$

with

$$\begin{aligned} Z_{\rm p} &= \frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\left[-\frac{\partial}{\partial E}\log(1+A_1)\right]e^{-\beta E}dE \\ &\quad+\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\left[-\frac{\partial}{\partial E}\log(1+A_2)\right]e^{-\beta E}dE+\frac{1}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\left[-\frac{\partial}{\partial E}\log\left(1+A_1^{-1}\right)\right]e^{-\beta E}dE\,, \end{aligned} \tag{111}$$

$$\begin{aligned} Z_{\rm np} &= \frac{\beta}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\sum_{n=1}^{\infty}\frac{1}{n}(-1)^n\left[B\left(\frac{1}{D^+_{A_1}D^\pm_{A_2}}+\frac{1}{D^-_{A_1}D^\pm_{A_2}}\right)+\frac{B^2}{D^+_{A_1}D^-_{A_1}D^\pm_{A_2}}\right]^n \\ &\simeq \frac{\beta}{2\pi i}\int_{\epsilon-i\infty}^{\epsilon+i\infty}\sum_{n=1}^{\infty}\frac{1}{n}(-1)^n\left[e^{\pm\pi i\frac{E}{\hbar\omega_{A_2}}}\left\{2\cos\left(\frac{E}{\hbar\omega_{A_1}}\right)\frac{B}{2\pi}\Gamma\left(\frac{1}{2}-\frac{E}{\hbar\omega_{A_1}}\right)\Gamma\left(\frac{1}{2}-\frac{E}{\hbar\omega_{A_2}}\right)\right.\right. \\ &\quad\left.\left.+\frac{B^2}{(2\pi)^{3/2}}\Gamma\left(\frac{1}{2}-\frac{E}{\hbar\omega_{A_1}}\right)^2\Gamma\left(\frac{1}{2}-\frac{E}{\hbar\omega_{A_2}}\right)\right\}\right]^n e^{-\beta E}dE\,. \end{aligned} \tag{112}$$