Semi-classics, adiabatic continuity and resurgence in quantum theories

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Some of the work presented here is done in collaboration with :
Yaffe, Shifman, Argyres, Poppitz, Schaefer, Dunne, Cherman, Sulejmanpasic, Tanizaki

0) Thanks for nice talks to Tatsu Misumi and Gerald Dunne

1) Critical points at infinity vs. real/complex bions in QM

2) Coupling a TQFT to QM

3) Adiabatic continuity and deformed Yang-Mills

4) Coupling a TQFT to YM

5) Critical points at infinity and magnetic/neutral bions in QFT

I will review some ideas and some new, and will tell you two parallel stories with the hope to merge them. Will also describe some (yet unresolved) puzzles.

Part 1

Critical points at infinity and real/complex bions in QM

QM with Grassmann valued fields

I will consider first the following QM systems.
(Many parallels with the saddles in semi-classical QFTs with fermions.)

$$S = \frac{1}{g}\int dt \left(\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}(W')^2 + \tfrac{1}{2}(\bar{\psi}_i\dot{\psi}_i - \dot{\bar{\psi}}_i\psi_i) + \tfrac{1}{2}W''[\bar{\psi}_i, \psi_i]\right) , \qquad i = 1, \ldots, N_f \, .$$

$N_f = 1$ SUSY QM

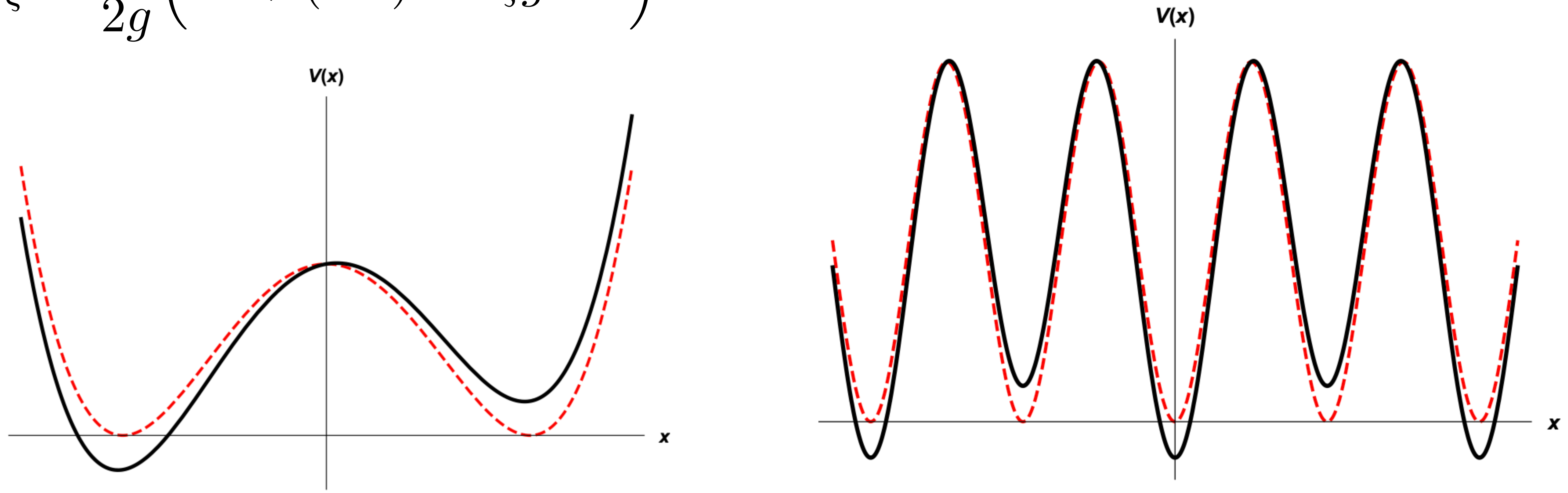
$N_f > 1$ related to QES systems. If exp[+W] or exp[-W] is normalizable, the lowest N_f states are exactly solvable! These systems are called Quasi-Exactly Solvable (QES) (Turbiner, Ushveridze 87), and to my mind, not less interesting than supersymmetric QM (very likely more.)

Quantizing the fermions, (or integrating them out exactly), we end up with

$$\widehat{H} = \bigoplus_{k=0}^{N_f} \deg(\mathcal{H}_{(N_f,k)})\widehat{H}_{(N_f,k)}, \qquad \deg(\mathcal{H}_{(N_f,k)}) = \binom{N_f}{k}$$

$$\hat{H}_{(N_f,k)} = \frac{g}{2}\,\hat{p}^2 + \frac{1}{2g}\Big((W')^2 + \zeta g W''\Big) \qquad \zeta = 2k - N_f, \qquad\qquad k = 0, \dots, N_f\,.$$

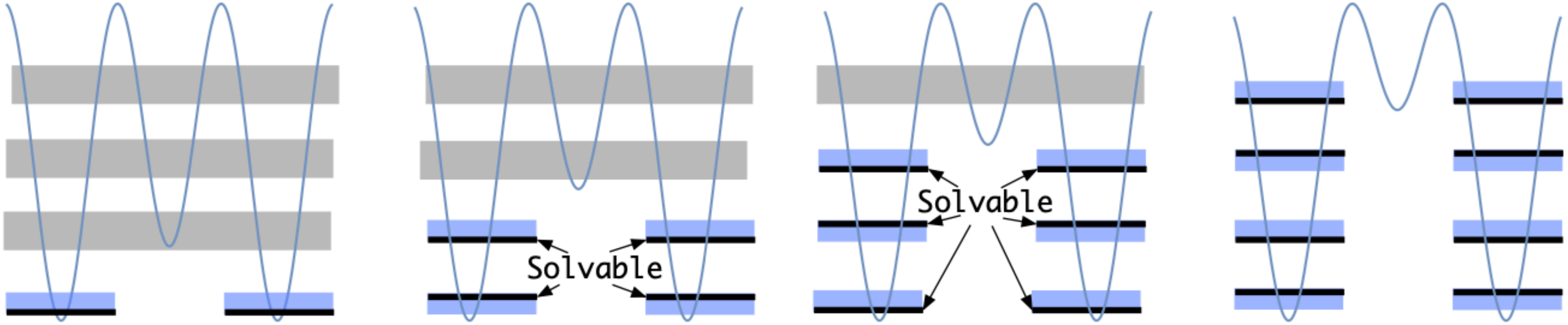
$$\mathcal{L}_\zeta = \frac{1}{2g}\Big(\dot{x}^2 + (W')^2 \pm \zeta g W''\Big)$$



Note that the potential has a classical and quantum part. The tilting is a one-loop quantum effect, induced by integrating out fermions.

If the tilting is rendered classical, the story changes quite a bit. But such quantum induced potential appears naturally by integrating out fermions both in QM and QFT, it is worthwhile to discuss this system for its own right.

Examples of exactly solvable states



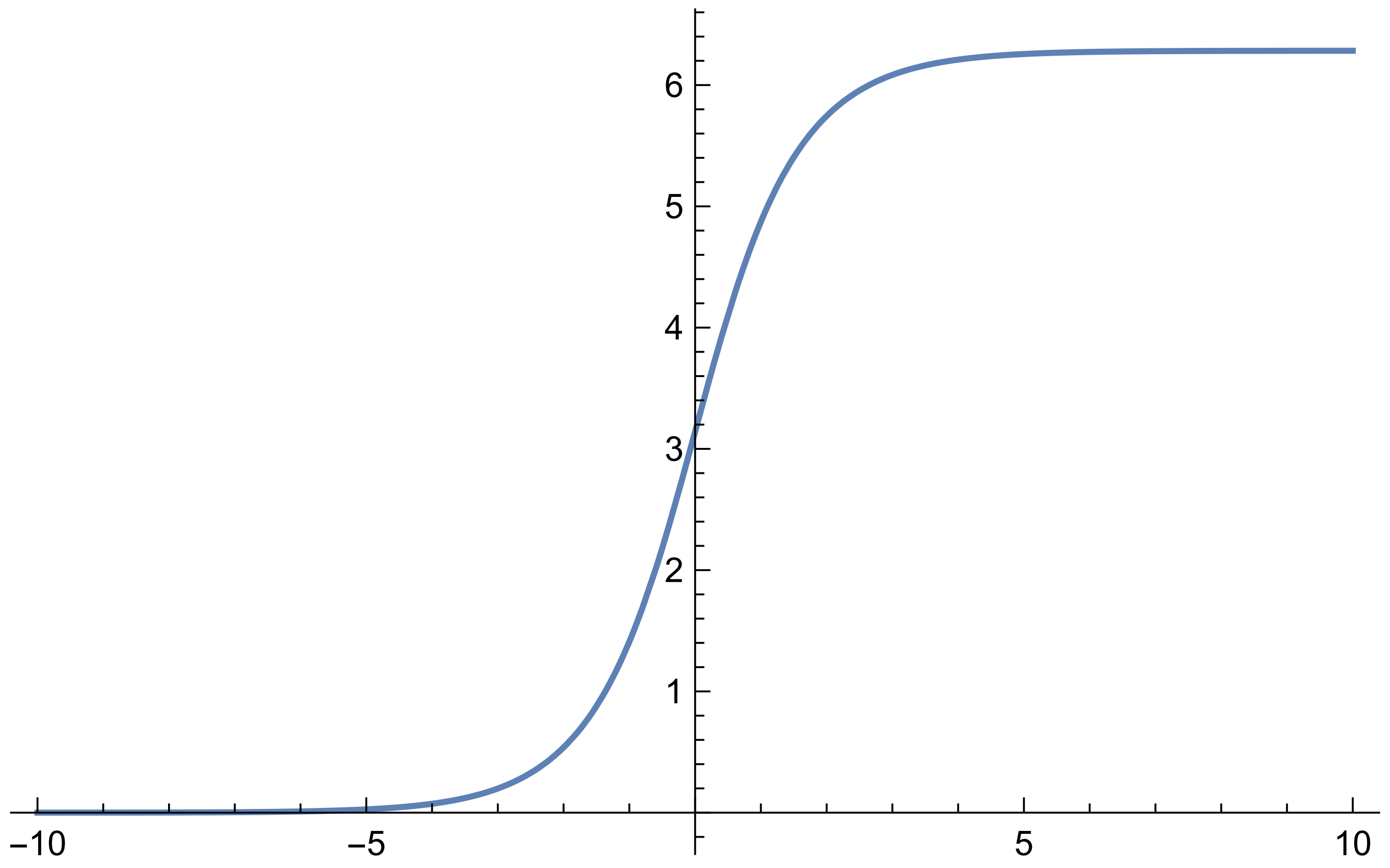
$\zeta = 1$ $\zeta = 2$ $\zeta = 3$ $\zeta = 4$

Basics of instantons- 1

$$\int \tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}(W'(x))^2 = \int \tfrac{1}{2}\underbrace{(\dot{x} \mp W'(x))^2}_{\geq 0} \pm \dot{x}W' \geq \left|\int dW\right| = |W(x_2) - W(x_1)|$$

$$\dot{x} = \pm W'(x)\,.$$ Instanton equation

$$W(x) = 4\cos\left(\frac{x}{2}\right) \Longrightarrow x_I(\tau) = 4\arctan\left(\exp[\tau - \tau_c]\right)\,,$$ Instanton solution



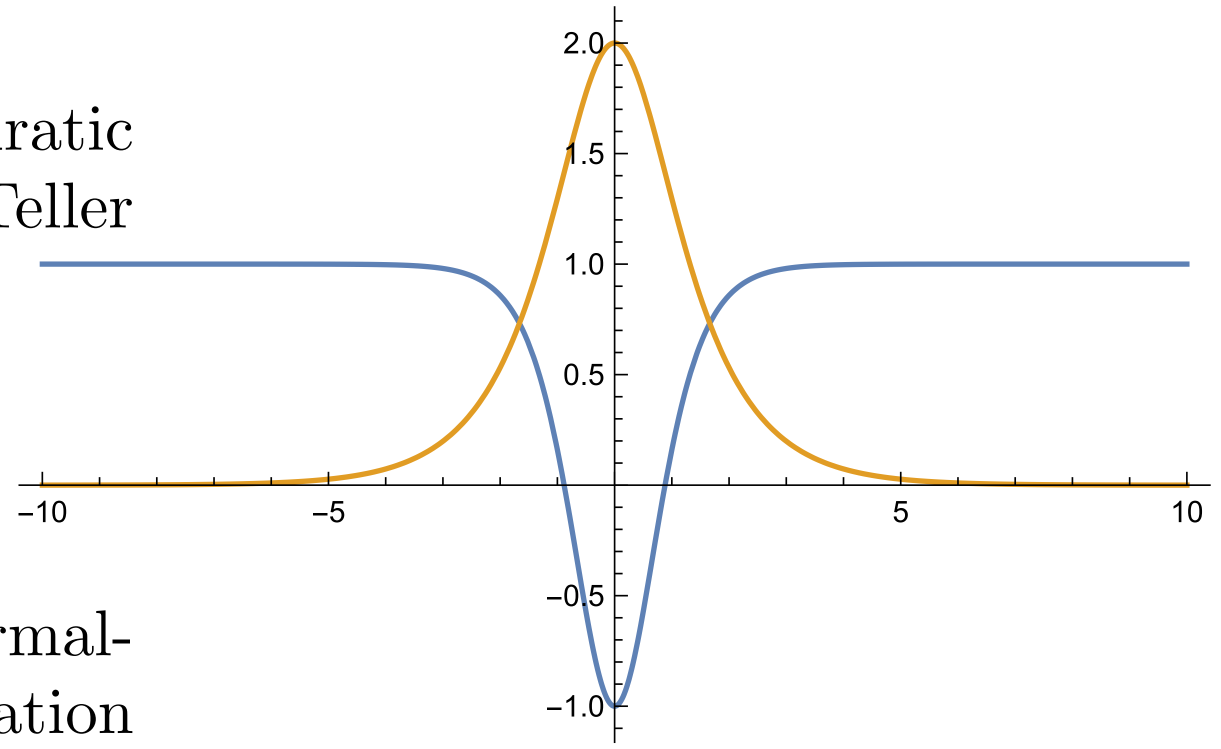
Basics of instantons-2

The instanton amplitude:

$$\mathcal{I} \equiv \xi = J_{\tau_c}\, e^{-S_I} \left[\frac{\widehat{\det}\,\mathcal{M}_I}{\det \mathcal{M}_0}\right]^{-\frac{1}{2}} P_I(g)\,,$$

- The overall amplitude: density of the instantons. Characteristic separation between instantons: $\sim e^{+S_I}$, dilute instanton gas.
- $J_{t_c} = \sqrt{S_I/(2\pi)}$: Jacobian associated with the bosonic zero mode.
- $\mathcal{M}_I = -\frac{d^2}{d\tau^2} + V''(x)|_{x=x_I(t)} = -\frac{d^2}{d\tau^2} + 1 - 2\,\mathrm{sech}^2(\tau-\tau_c)\,,$: quadratic fluctuation operator in the background of the instanton. (Pöschl-Teller form). Exact zero mode is given by

$$\Psi_0(\tau) = \dot{x}_I(\tau) = \frac{2}{\cosh(\tau-\tau_c)}$$



The "hat": the zero mode has to be removed, and $\det \mathcal{M}_0$ is a normalization factor, which we take to be the corresponding free fluctuation operator.

- Perturbative expansion around instanton:

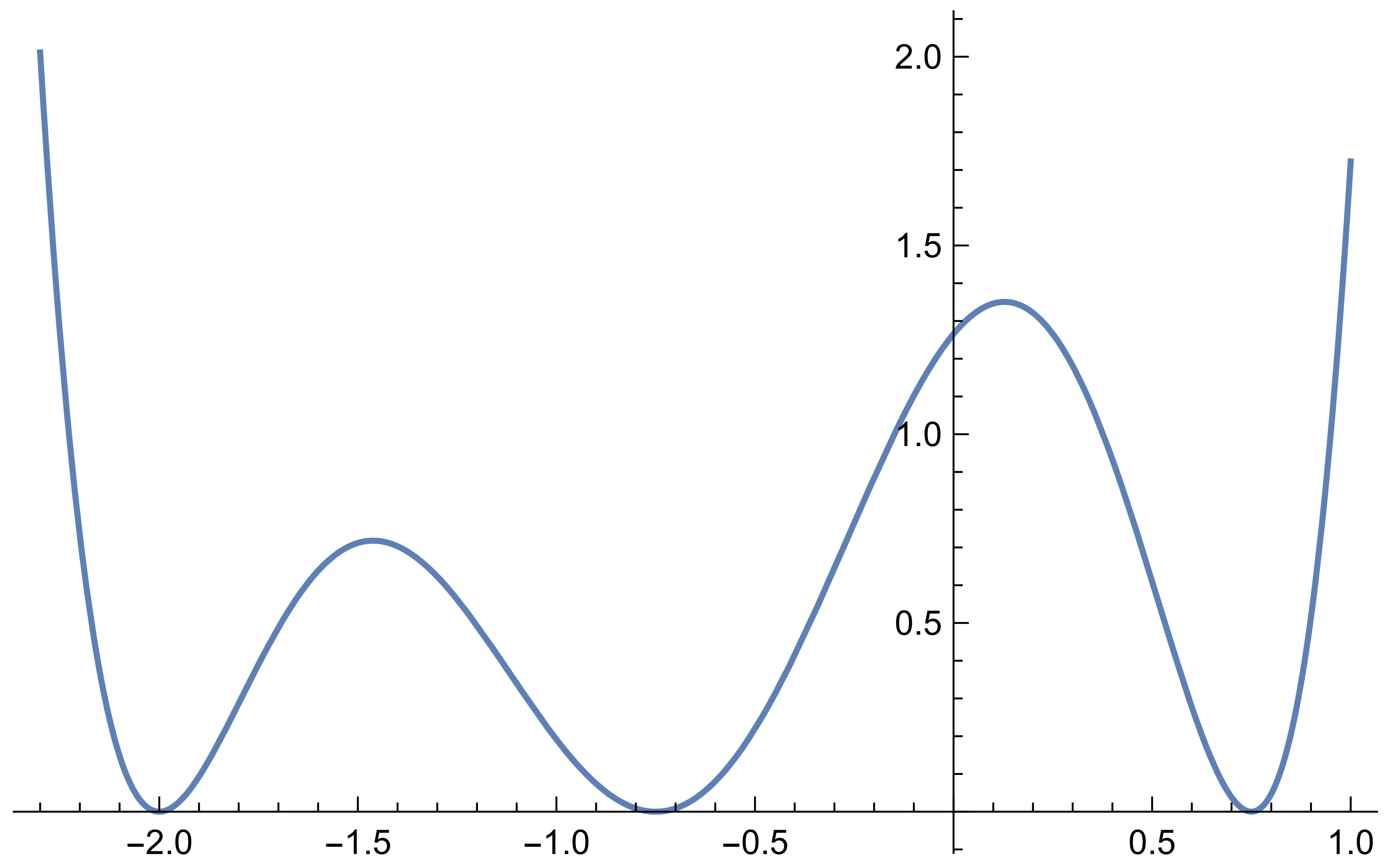
$$P_I(g) = \sum_{n=0}^{\infty} b_{I,n} g^n,$$

which is a formal asymptotic series, which is in general not Borel summable.

- The determinant of the instanton fluctuation operator can be computed using the Gel'fand-Yaglom (GY) method. (See Marino's book).

Remark: Do instantons always contribute to physical observables?

In almost all books and texts, you will see the discussion of double-well or periodic potential, but not a more generic potential with harmonic degenerate minima as shown in figure. Why not?



Despite the fact that there are exact instanton solutions, for generic potential of this type, they typically do not contribute to the spectrum at the exp[-S] order, rather, the first NP contribution appears at order exp[-2S], related to the concept of critical point at infinity (which I will explain).

The reason instantons do not contribute at leading order is that the determinant of fluctuation operator is infinite unless the frequency in two consecutive well are the same.

Therefore, in QM, instanton contributing to spectrum is exception instead of being a rule.

Perturbation theory by Bender-Wu method

Bender-Wu Mathematica package written by Tin Sulejmanpasic:
https://library.wolfram.com/infocenter/MathSource/9479/.

Description

The BenderWu package allows for analytic computation of the perturbative series in 1D quantum mechanics around a harmonic minimum of the potential. The code is based on the method pioneered by Bender and Wu.

$$
\begin{aligned}
E^{\rm pert}(N,g) &\sim \sum_{n=0}^{\infty} \hbar^n a_n(N) \\
&\sim \left[N+\frac{1}{2}\right] - \frac{g}{16}\left[\left(N+\frac{1}{2}\right)^2+\frac{1}{4}\right] - \frac{g^2}{16^2}\left[\left(N+\frac{1}{2}\right)^3+\frac{3}{4}\left(N+\frac{1}{2}\right)\right] \\
&\quad - \frac{g^3}{16^3}\left[\frac{5}{2}\left(N+\frac{1}{2}\right)^4+\frac{17}{4}\left(N+\frac{1}{2}\right)^2+\frac{9}{32}\right] \\
&\quad - \frac{g^4}{16^4}\left[\frac{33}{4}\left(N+\frac{1}{2}\right)^5+\frac{205}{8}\left(N+\frac{1}{2}\right)^3+\frac{405}{64}\left(N+\frac{1}{2}\right)\right] - \dots
\end{aligned}
$$

$$
a_n(N) \sim -\frac{2^{2N}}{\pi\,(N!)^2}\frac{\Gamma(n+2N+1)}{(2S_I)^{n+2N+1}}
$$

Large-order factorial growth for harmonic level N

$$
a_n(N=0) \sim -\frac{1}{\pi}\frac{n!}{(2S_I)^{n+1}}\left(1-\frac{5}{2}\cdot\frac{(2S_I)^1}{n}-\frac{13}{8}\cdot\frac{(2S_I)^2}{n(n-1)}+\dots\right)
$$

Large-order factorial growth for ground state.

Instanton interactions

Since instanton equations and Euclidean eq of motion are non-linear, two instanton configurations is not a solution at finite separation.

$$x_{\mathcal{I}\mathcal{I}}(\tau) = x_{\mathcal{I}}(\tau - \tau_1) + x_{\mathcal{I}}(\tau - \tau_2),$$
$$x_{\mathcal{I}\bar{\mathcal{I}}}(\tau) = x_{\mathcal{I}}(\tau - \tau_1) - x_{\mathcal{I}}(\tau - \tau_2),$$

$$S_{\mathcal{I}\mathcal{I}}(\tau_{12}) = 2S_I + \frac{A}{g}e^{-\tau_{12}}, \qquad \text{repulsive,}$$
$$S_{\mathcal{I}\bar{\mathcal{I}}}(\tau_{12}) = 2S_I - \frac{A}{g}e^{-\tau_{12}}, \qquad \text{attractive}$$

Attractive/repulsive are just words, inheritance from old literature. Caused too much confusion in past. This formula just means that these combos are not exact solution for finite separation. That is all. Tau direction is called quasi-moduli space.

Cluster expansion

For particle on a circle with unique minimum on the circle (for simplicity)

In the $\beta \to \infty$ limit, we can write Z as

$$Z = e^{-\beta E_0 P_0(g)} \left(1 + \frac{\xi}{1!} \int d\tau_1 + \frac{\xi^2}{2!} \int d\tau_1 d\tau_2 \; e^{-V_{12}} + \frac{\xi^3}{3!} \int d\tau_1 d\tau_2 d\tau_3 \; e^{-V_{123}} + \dots \right) .$$

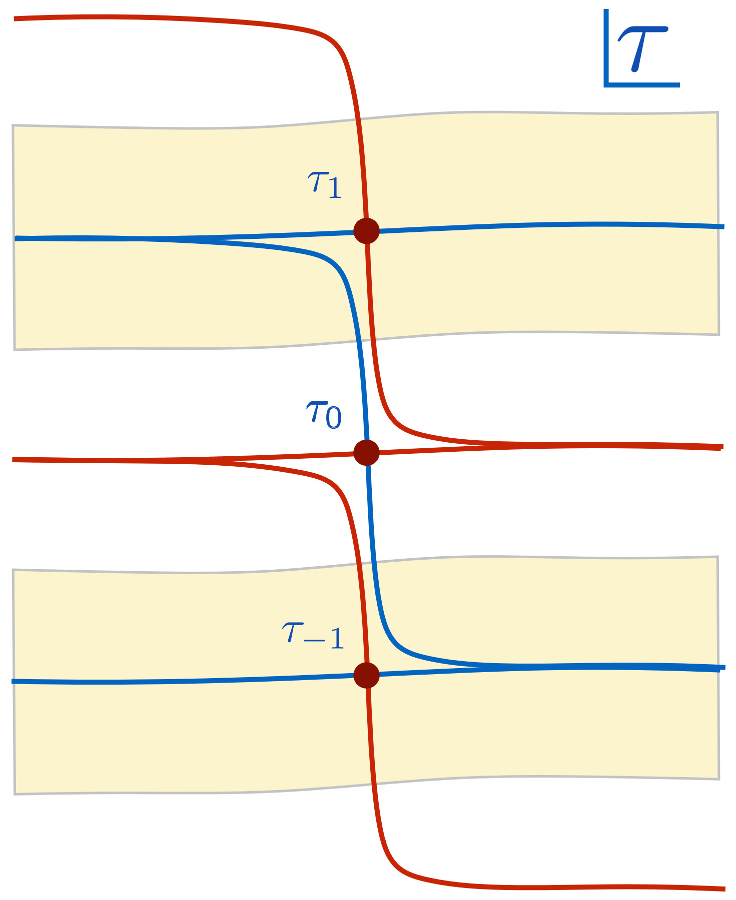
whee $\xi \sim e^{-S_I}$ is the instanton amplitude.

$$\begin{aligned} Z_{\text{dilute}} &= e^{-\beta\left(E_0 P_0(g) - [\mathcal{I}] - [\bar{\mathcal{I}}] - [\mathcal{I}^2] - [\bar{\mathcal{I}}^2] - [\mathcal{I}\bar{\mathcal{I}}]_\pm - [\bar{\mathcal{I}}\mathcal{I}]_\pm - [\mathcal{I}^3] - [\mathcal{I}^2\bar{\mathcal{I}}] \dots\right)} \\ &= e^{-\beta E_0 P_0(g)} \left(\sum_{n_\mathcal{I}=0}^{\infty} \frac{\beta^{n_\mathcal{I}} [\mathcal{I}]^{n_\mathcal{I}}}{n_\mathcal{I}!} \right) \left(\sum_{n_{\bar{\mathcal{I}}}=0}^{\infty} \frac{\beta^{n_{\bar{\mathcal{I}}}} [\bar{\mathcal{I}}]^{n_{\bar{\mathcal{I}}}}}{n_{\bar{\mathcal{I}}}!} \right) \left(\sum_{n_{\mathcal{I}\bar{\mathcal{I}}}=0}^{\infty} \frac{\beta^{n_{\mathcal{I}\bar{\mathcal{I}}}} [\mathcal{I}\bar{\mathcal{I}}]_\pm^{n_{\mathcal{I}\bar{\mathcal{I}}}}}{n_{\mathcal{I}\bar{\mathcal{I}}}!} \right) \dots \end{aligned}$$

Compactify $\mathbb{R} \to S^1_\beta$ in order to study $Z(\beta) = \operatorname{Tr}\left[e^{-\beta H}\right]$.
The interaction between two events is modified in a fairly obvious way into:

$$S(\tau) = \pm \frac{A}{g}\left(e^{-\tau} + e^{-(\beta-\tau)}\right)$$

$$[\mathcal{I}\bar{\mathcal{I}}] = \left(\frac{1}{2}\int_0^\beta d\tau\; e^{\frac{A}{g}\left(e^{-\tau}+e^{-(\beta-\tau)}\right)} - \beta/2\right)[\mathcal{I}][\bar{\mathcal{I}}]$$



The Lefschetz thimbles for the $\mathcal{I}\bar{\mathcal{I}}$ saddle, showing the downward flows (blue curves) connecting τ_0 to $\tau_{\pm 1}$ when $g \to g\,e^{i\theta}$ with $\theta \to 0^+$. The directions are flipped about the imaginary axis for $\theta \to 0^-$.

Borel-Ecalle summability in bosonic theory

$$[\mathcal{I}\bar{\mathcal{I}}]_{\pm} = \left(\mp i\pi \; - \gamma - \log\left(\frac{A}{g}\right) + \ldots\right)[\mathcal{I}][\bar{\mathcal{I}}]$$

$$[\mathcal{I}\bar{\mathcal{I}}]_{\pm} \sim \left(\mp i\pi \; - \gamma - \log\left(\frac{A}{g}\right) + \ldots\right) e^{-(2S_I)/g}\left(1 - \frac{5}{2}\cdot g - \frac{13}{8}\cdot g^2 \ldots\right)$$

$$a_n(N=0) \sim -\frac{1}{\pi}\frac{n!}{(2S_I)^{n+1}}\left(1 - \frac{5}{2}\cdot\frac{(2S_I)^1}{n} - \frac{13}{8}\cdot\frac{(2S_I)^2}{n(n-1)} + \ldots\right)$$

$$\mathrm{Im}\,\mathbb{B}_{0,\theta=0^{\pm}} + \mathrm{Im}\,[\mathcal{I}\bar{\mathcal{I}}]_{\theta=0^{\pm}} = 0\,, \qquad \text{up to } O(e^{-4S_I})$$

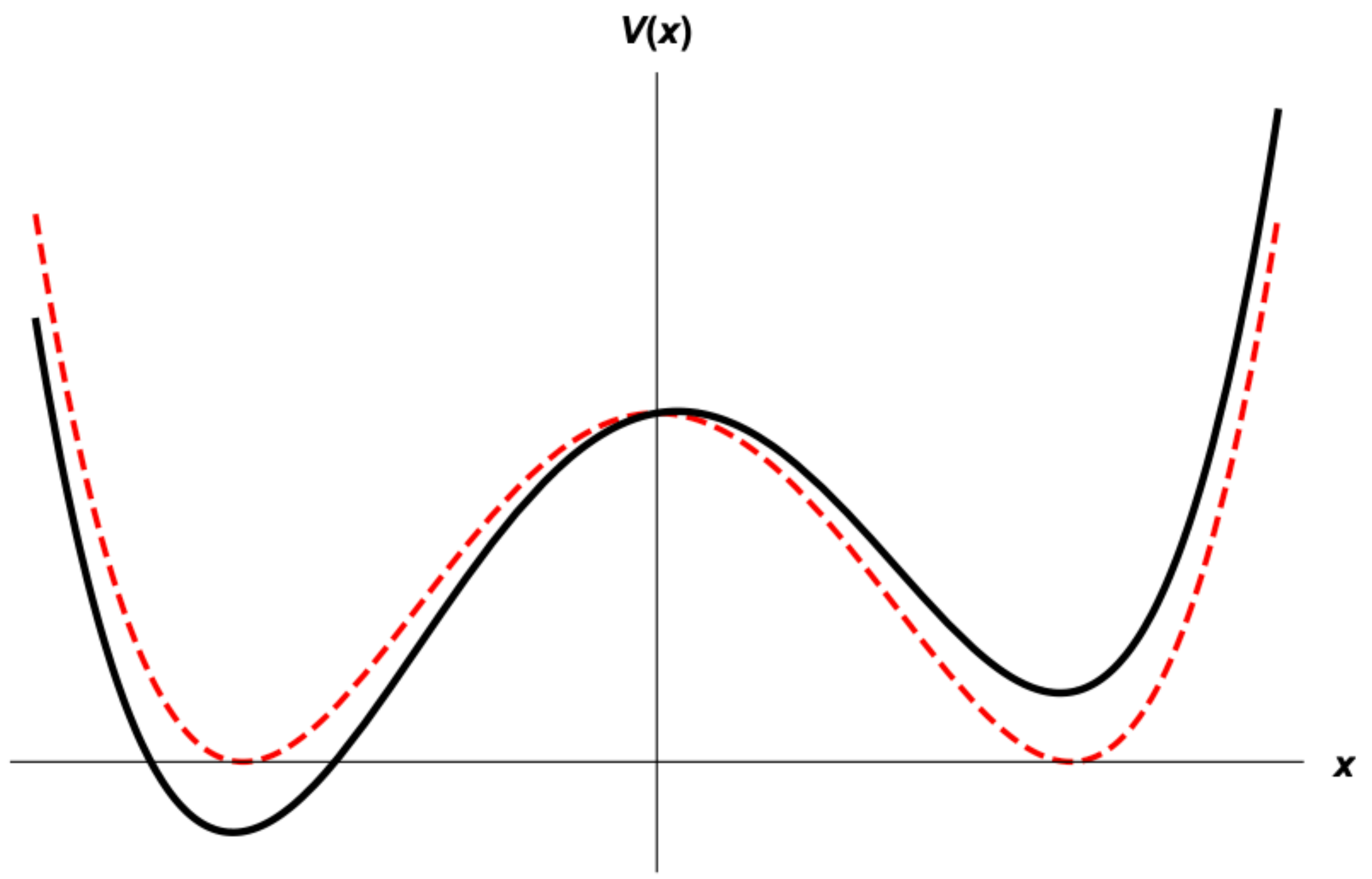
The leading terms (structures) obtained in Bogomolny and Zinn-Justin early 80s, but not sufficiently appreciated. The interesting thing is, B-ZJ was not an unknown work. The problem was that their methods in the derivation did not sufficiently convince people. (Otherwise, they would held this conference in ~1985). I was personally fascinated by what they did, and was convinced that their main claim was correct.

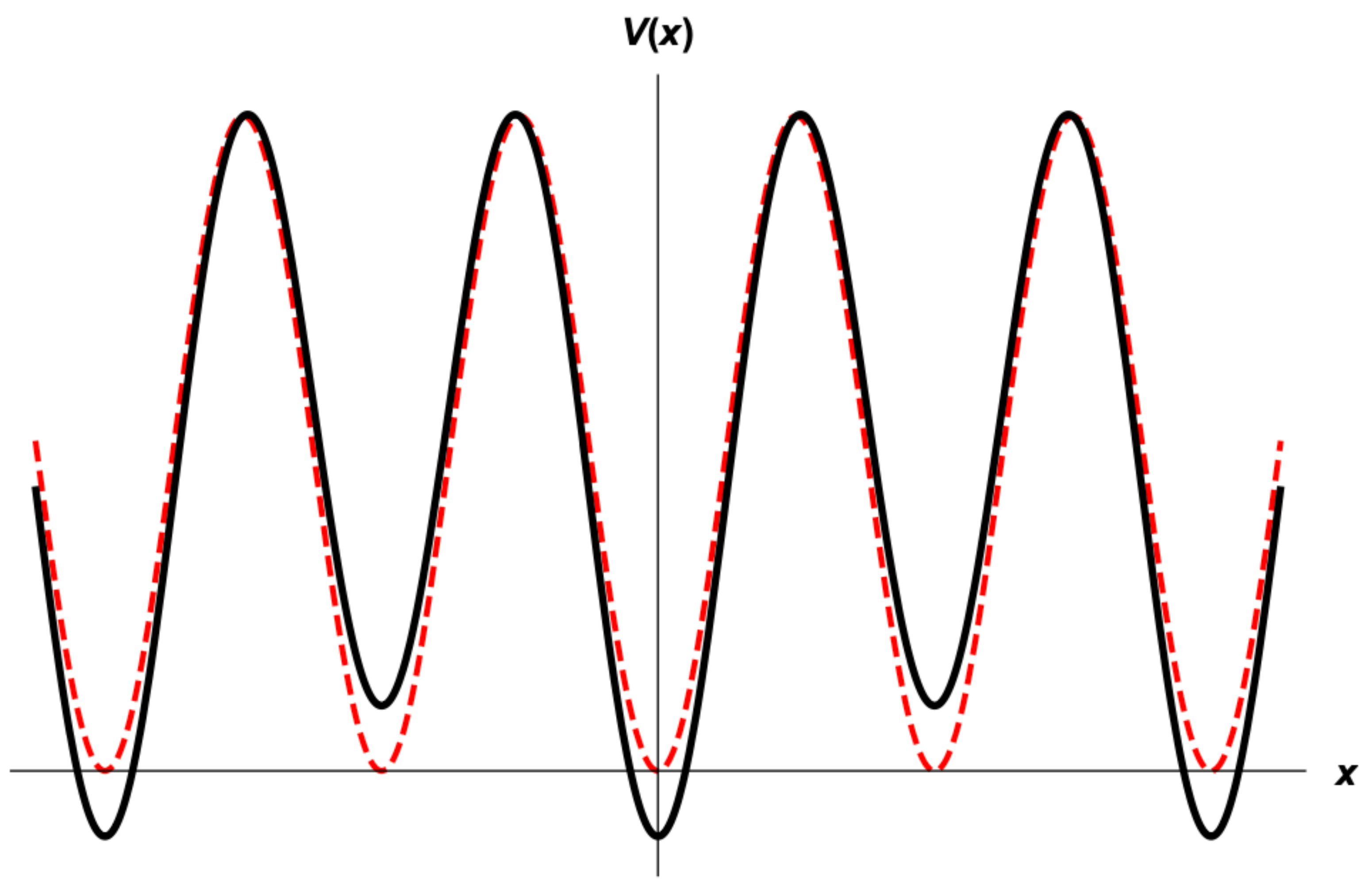
The overall structure was obtained in 2014, in Gerald Dunne and MU.

SUSY, QES and in between: parametric resurgence

$$S = \frac{1}{g}\int dt \left(\tfrac{1}{2}\dot{x}^2 + \tfrac{1}{2}(W')^2 + \tfrac{1}{2}(\bar{\psi}_i\dot{\psi}_i - \dot{\bar{\psi}}_i\psi_i) + \tfrac{1}{2}W''[\bar{\psi}_i, \psi_i]\right) , \qquad i = 1, \ldots, N_f \, .$$

$$\mathcal{L}_\zeta = \frac{1}{2g}\left(\dot{x}^2 + (W')^2 \pm \zeta g W''\right)$$





Instanton interactions in the presence of fermions or quantum tilting

$$S_{\mathcal{I}\mathcal{I}}(\tau) = \overbrace{+\frac{A}{g}\left(e^{-\tau} + e^{-(\beta-\tau)}\right)} + \overbrace{\zeta\,\tau} \, .$$

$$S_{\mathcal{I}\bar{\mathcal{I}}}(\tau) = \underbrace{-\frac{A}{g}\left(e^{-\tau} + e^{-(\beta-\tau)}\right)}_{\text{classical}} + \underbrace{\zeta\,\tau}_{\text{quantum}}$$

Concept of critical point at infinity and non-Gaussian critical points

$$I_+(\zeta,g) \equiv \int_{\Gamma^{\theta=0^\pm}_{\mathrm{QZM}}} d\tau\, e^{-\frac{A}{g}\left(e^{-\tau}+e^{-(\beta-\tau)}\right)} e^{-\zeta\tau}$$

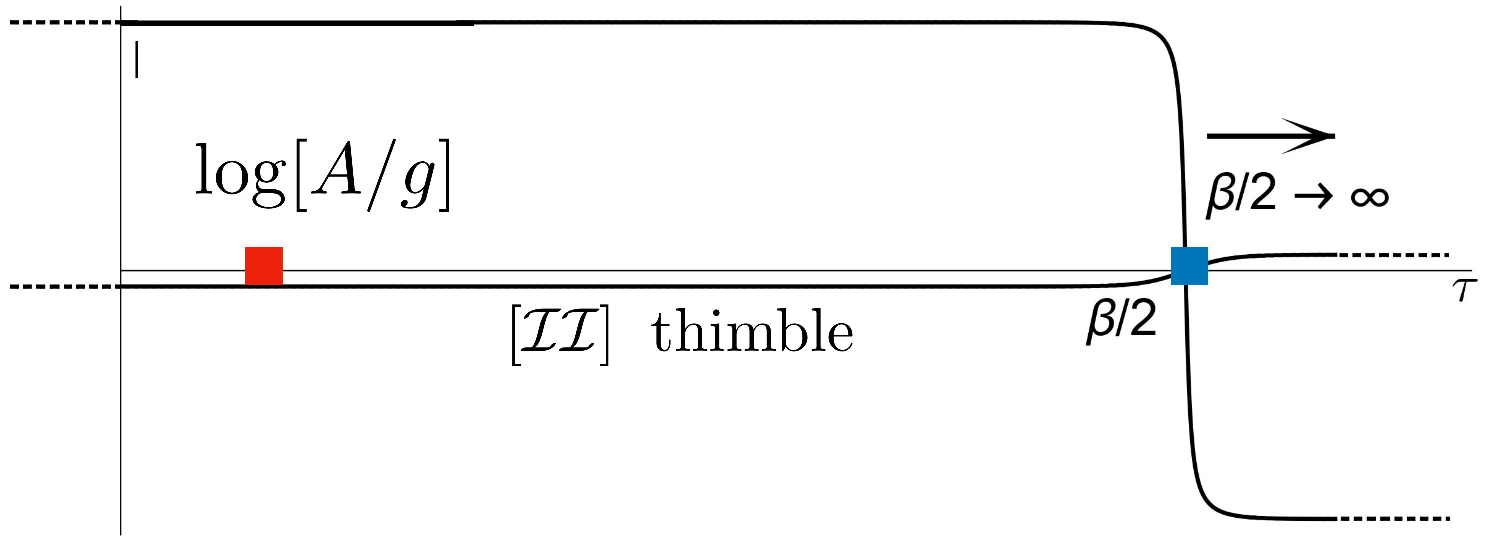
$$I_-(\zeta,g) \equiv \int_{\Gamma^{\theta=0^\pm}_{\mathrm{QZM}}} d\tau\, e^{\frac{A}{g}\left(e^{-\tau}+e^{-(\beta-\tau)}\right)} e^{-\zeta\tau}$$

$$\lim_{\beta\to\infty} e^{-\frac{2A}{g}\left(e^{-\beta/2}\right)} e^{-\zeta\beta/2} = 0\,.$$

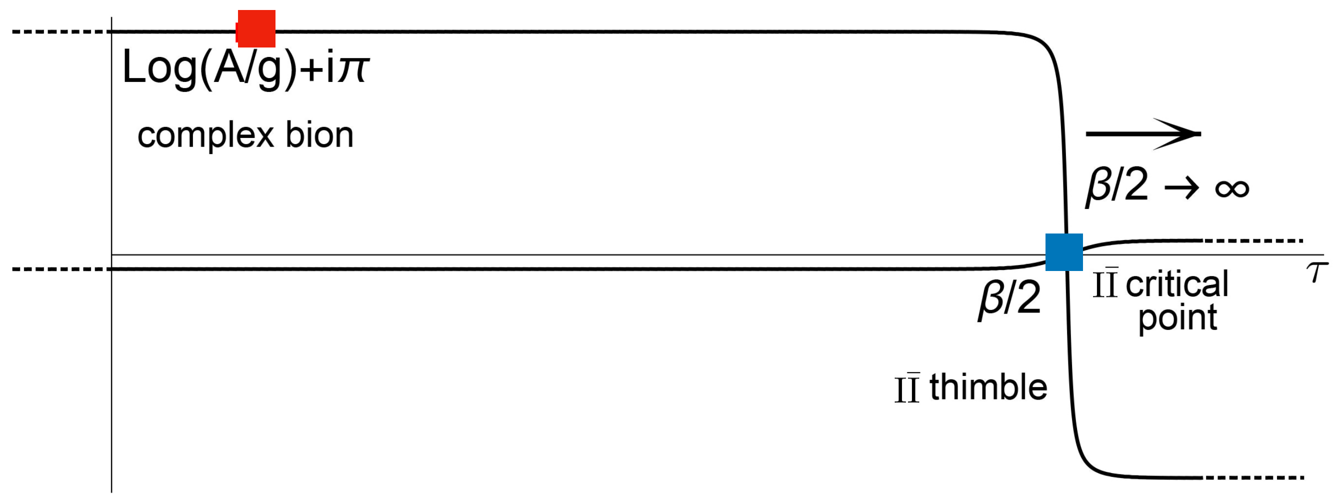
$$\lim_{\beta\to\infty} e^{\frac{2A}{g}\left(e^{-\beta/2}\right)} e^{-\zeta\beta/2} = 0$$

$$\begin{aligned}[\mathcal{I}\mathcal{I}] &= I_+(\zeta,g) \times [\mathcal{I}]^2 \\ &= \left(\frac{g}{A}\right)^{\zeta} \Gamma(\zeta) \times \frac{S_I}{2\pi}\left[\frac{\widehat{\det}\,\mathcal{M}_I}{\det \mathcal{M}_0}\right]^{-1} e^{-2S_I} \\ &= \frac{1}{2\pi}\left(\frac{g}{32}\right)^{\zeta-1}\Gamma(\zeta) e^{-2S_I}\,.\end{aligned}$$

$$\begin{aligned}[\mathcal{I}\bar{\mathcal{I}}]_\pm &= I_-(\zeta,g) \times [\mathcal{I}]^2 \\ &= e^{\pm i\pi\zeta}\left(\frac{g}{A}\right)^{\zeta} \Gamma(\zeta) \times \frac{S_I}{2\pi}\left[\frac{\widehat{\det}\,\mathcal{M}_I}{\det \mathcal{M}_0}\right]^{-1} e^{-2S_I} \\ &= \frac{1}{2\pi}\left(\frac{g}{32}\right)^{\zeta-1}\Gamma(\zeta) e^{-2S_I} e^{\pm i\pi\zeta}\,.\end{aligned}$$



$[\mathcal{I}\bar{\mathcal{I}}]$ thimble

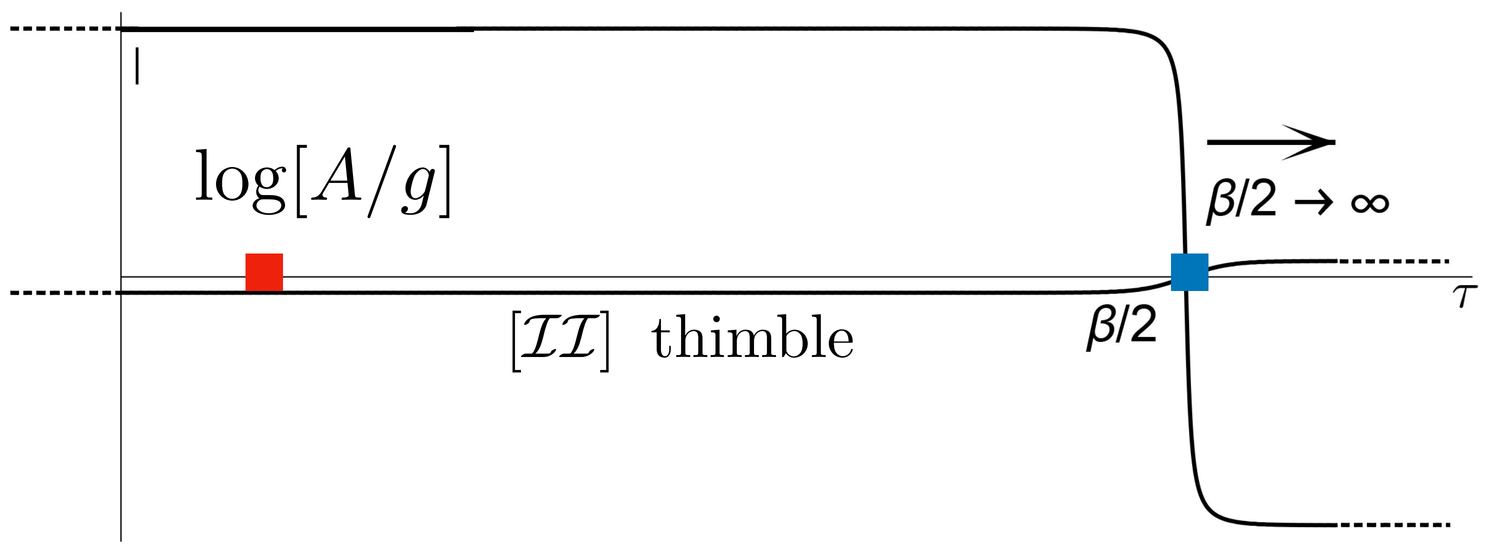


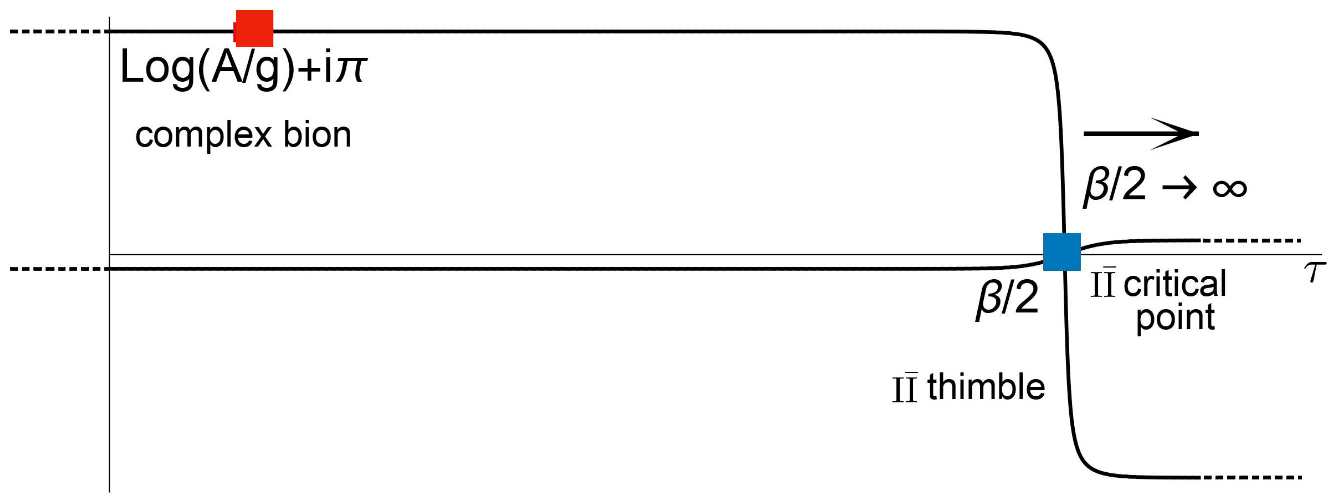
Concept of critical point at infinity and non-Gaussian critical points

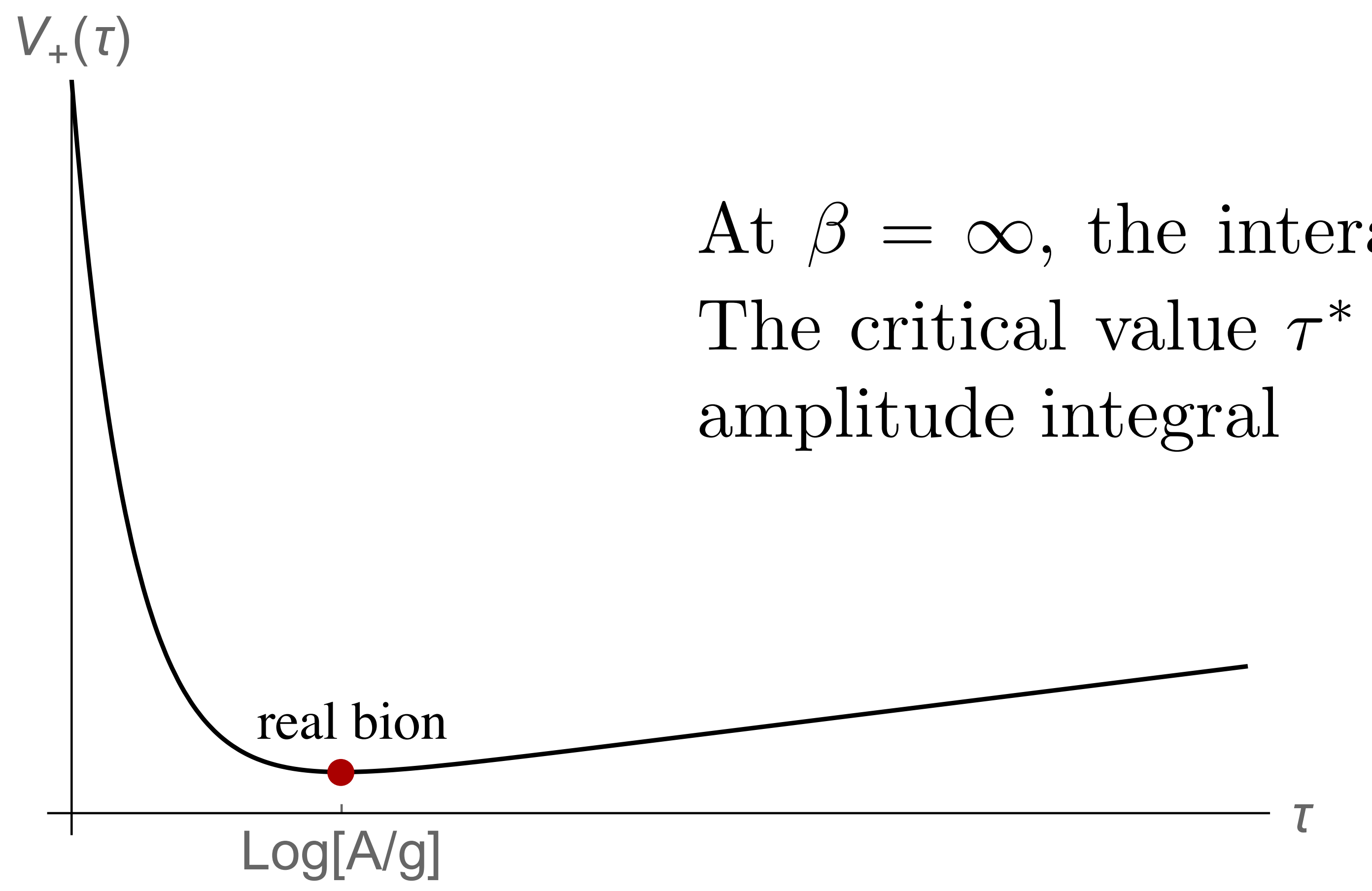
Unlike Gaussian critical point, the critical point at infinity itself does not contribute. However, its thimble gives major contribution.

The major contribution on the thimble comes about from configurations (bions) which are exact solutions to quantum modified holomorphic equations of motions. The equations are for a holomorphic classical mechanical systems, and holomorphic version of Newton's equations. These are called real and complex bions and I will show you their plots.

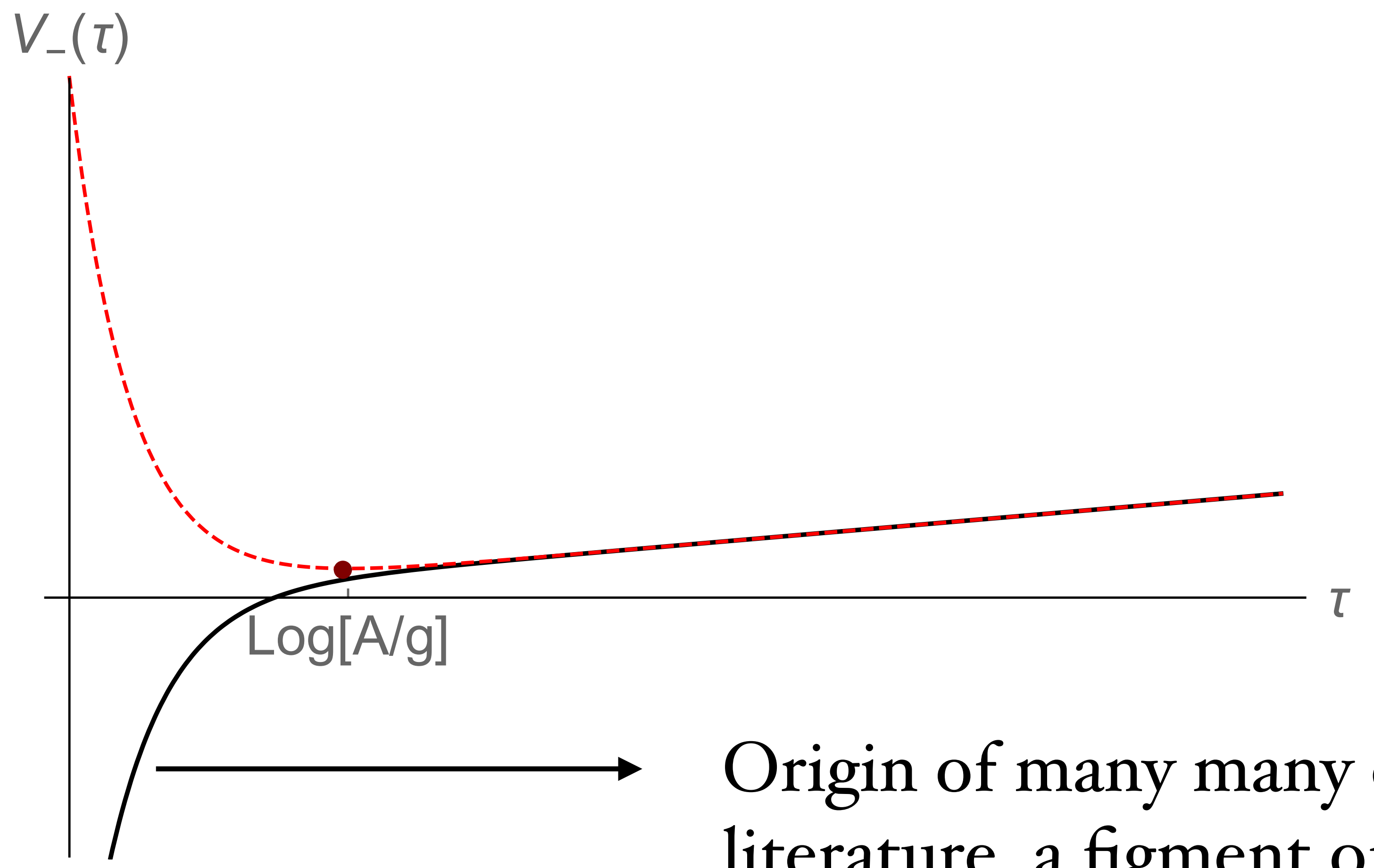
$$\frac{d^2 z}{dt^2} = \frac{\partial V}{\partial z} \qquad \text{or equivalently} \qquad \frac{d^2 x}{dt^2} = +\frac{\partial V_{\rm r}}{\partial x}\,, \qquad \frac{d^2 y}{dt^2} = -\frac{\partial V_{\rm r}}{\partial y}\,,$$







At $\beta = \infty$, the interaction potential between $\mathcal{I}$ and $\mathcal{I}$ is $V(\tau) = \frac{A}{g} e^{-\tau} + \zeta\, \tau$. The critical value $\tau^* = \ln{(A/g\,\zeta)}$ gives the dominant contribution to the $[\mathcal{I}\mathcal{I}]$ amplitude integral



Origin of many many confusions in literature, a figment of imagination.

(black-solid curve): For real values of the separation $\tau \in \mathbb{R}^+$, which is the naive (or customary) integration cycle, the interactions are completely attractive, and configuration is viewed as unstable. (red-dashed curve): the effective potential on the thimble. The value $\tau^* = \ln{(A/g\,\zeta)} + i\pi$ gives the dominant contribution to the $[\mathcal{I}\bar{\mathcal{I}}]$ amplitude integral.

Parametric resurgence
Working of resurgence at arbitrary ζ

$$
\begin{aligned}
E^{\text{pert}}(N,g;\zeta) &\sim \sum_{n=0}^{\infty} a_n(N;\zeta) g^n \\
&\sim \left(N+\frac{1}{2}-\frac{\zeta}{2}\right)+\frac{1}{8}\Big(-\left[2N^2+2N+1\right]+\left[2N+1\right]\zeta\Big)g \\
&+\frac{1}{64}\Big(-\left[4N^3+6N^2+6N+2\right]+\left[6N^2+6N+3\right]\zeta-\left[2N+1\right]\zeta^2\Big)g^2 \\
&+\frac{1}{256}\Big(-\left[10N^4+20N^3+32N^2+22N+6\right]+\left[20N^3+30N^2+32N+11\right]\zeta \\
&-\left[12N^2+12N+6\right]\zeta^2+\left[2N+1\right]\zeta^3\Big)g^3+\dots
\end{aligned}
$$

Thanks to Tin Sulejmanpasic for his BenderWu Mathematica package, this is possible as a symbolic calculation.

Large-order behavior can be extracted: (Kozcaz, Sulejmanpasic, Tanizaki, MU, 2016)

$$
\begin{aligned}
a_n(N=0;\zeta) &\sim -\frac{1}{\pi}\frac{1}{(8)^{\zeta-1}}\frac{1}{\Gamma(1-\zeta)}\frac{(n-\zeta)!}{(S_b)^{n-\zeta+1}} \\
&\times\left(b_0(\zeta)+\frac{(S_b)\,b_1(\zeta)}{n-\zeta}+\frac{(S_b)^2\,b_2(\zeta)}{(n-\zeta)(n-\zeta-1)}+\dots\right)
\end{aligned}
$$

Parametric resurgence

Where b's are non-trivial polynomials of zeta.

$$
\begin{aligned}
b_0(\zeta) &= 1 \\
b_1(\zeta) &= \frac{1}{8}\left(-5+5\zeta-\zeta^2\right) \\
b_2(\zeta) &= \frac{1}{128}\Big(-13+2\zeta+15\zeta^2-8\zeta^3+\zeta^4\Big), \\
b_3(\zeta) &= \frac{1}{3072}\Big(-\zeta^6+9\zeta^5-10\zeta^4-51\zeta^3-10\zeta^2+381\zeta-357\Big)
\end{aligned}
$$

And using the NP contributions to the energy:

$$E_{\pm}^{\text{n.p.}}(N=0,g;\zeta) \sim -(2[\mathcal{RB}]+2[\mathcal{CB}]_{\pm})$$

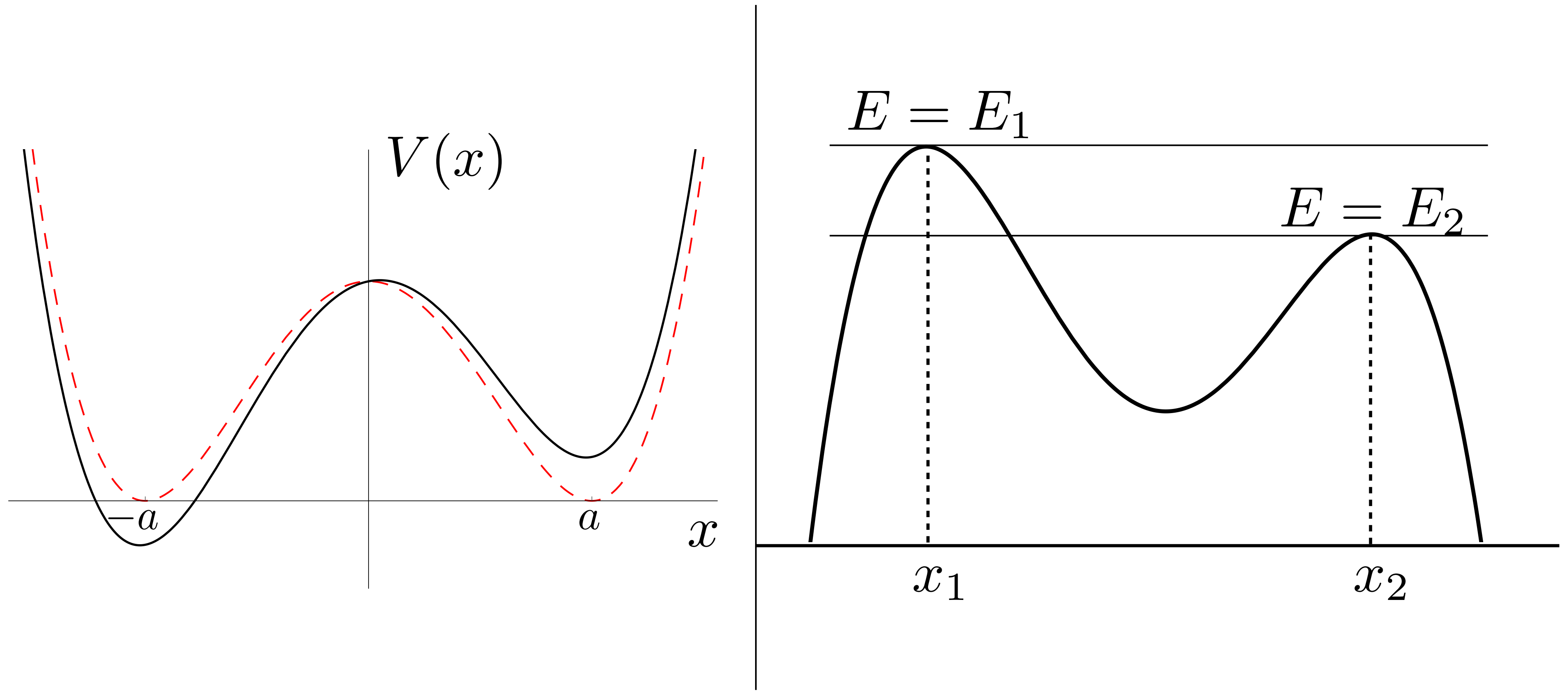
$$\sim \frac{1}{\pi}\left(\frac{g}{8}\right)^{\zeta-1}\Gamma(\zeta)(-1-e^{\pm i\pi\zeta})e^{-S_b/g}\underbrace{\left(b_0(\zeta)+b_1(\zeta)g+b_2(\zeta)g^2+b_3(\zeta)g^3+\dots\right)}_{\mathcal{P}_{\text{fluc}}(N=0,g;\zeta)}$$

$$\text{Im}\left[\mathcal{S}_{\pm}E^{\text{pert.}}(N=0,g,\zeta)+[\mathcal{CB}]_{\pm}(N=0,g,\zeta)\right]=0.$$

Quite remarkable, traditional form of resurgence.
At integer zeta, ambiguity disappears, pert th becomes convergent. We will find similar structure in QCD(adj) as a function of Nf.

Supersymmetric QM and complex bions-I

Take Double-well susy QM. This system breaks susy spontaneously. (Witten, 81)
Quantize fermions and reduce the system to Bose-Fermi pair of Hamiltonians with tilted potential.



$$V_{\pm} = \frac{1}{2}(z^2 - 1)^2 \pm gz$$

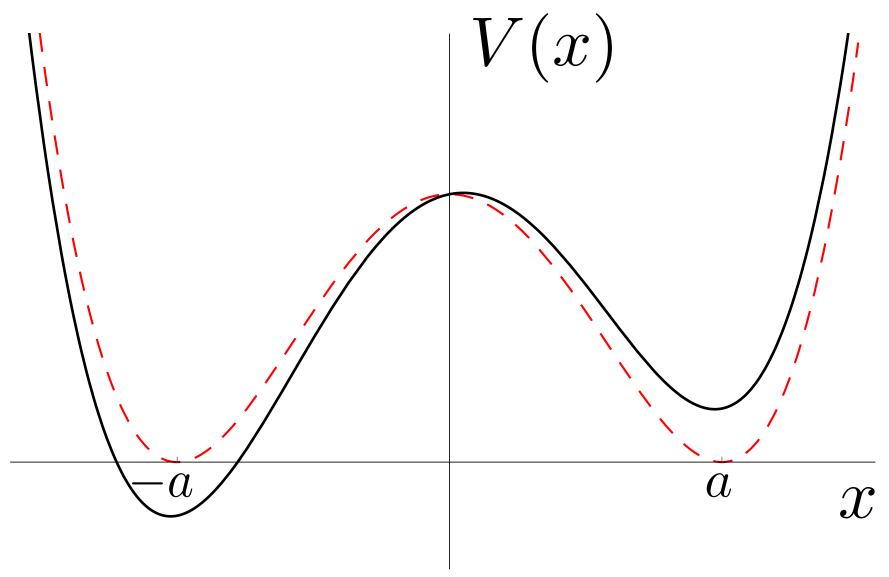
Ground state energy is zero to all orders in P.T. But is known to be lifted non-perturbatively. What causes it?

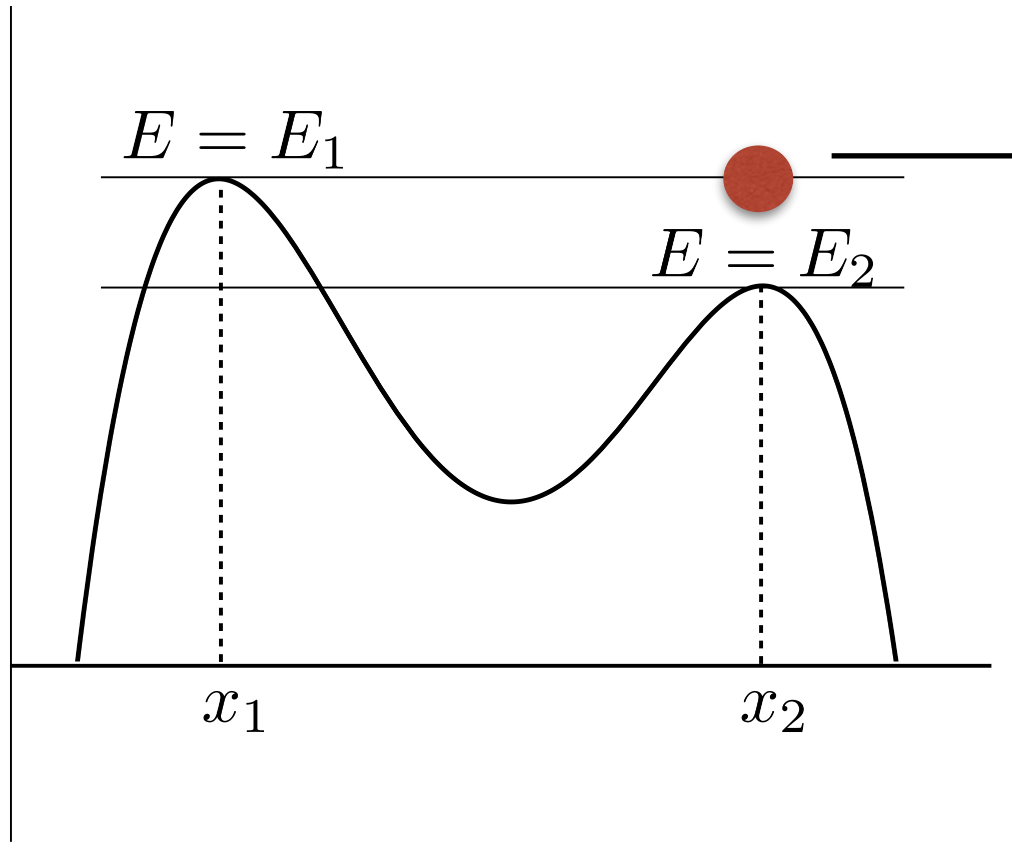
In the inverted potential, there is an obvious real bounce solution, but this is not related to ground state properties.

At level E1, the classical particle will fly of to infinity, infinite action, irrelevant. So, what causes the non-zer
energy in bosonized description?

Supersymmetric QM and necessity of complex bions!

Take Double-well susy QM. This system breaks susy spontaneously. (Witten, 81)
Quantize fermions and reduce the system to Bose-Fermi pair of Hamiltonians with tilted potential.



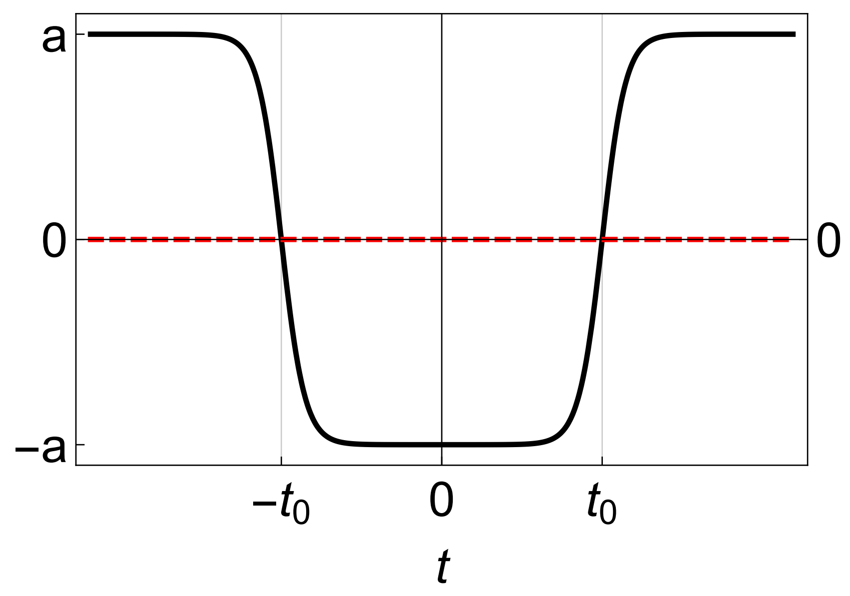


Complex conjugate turning points

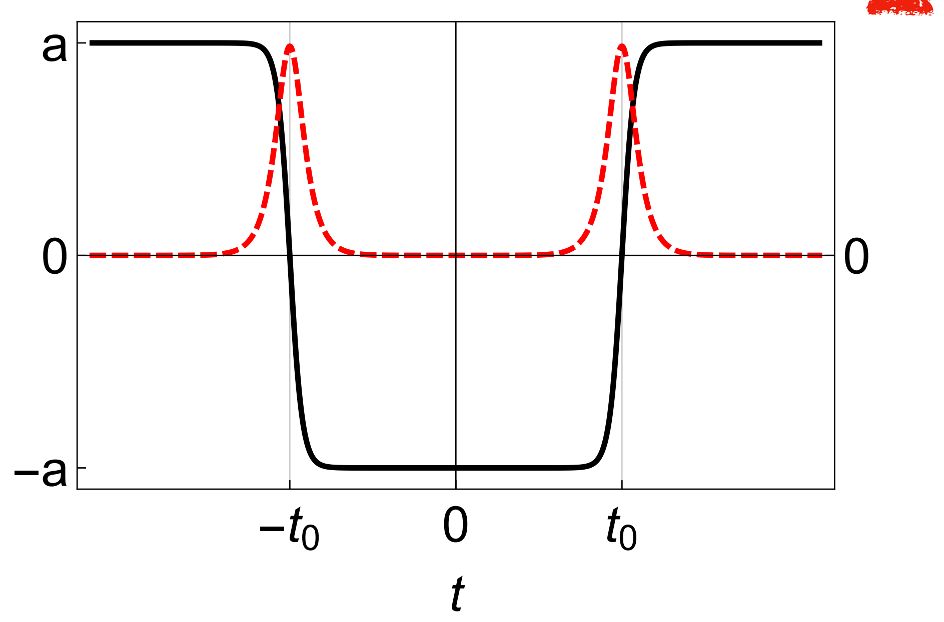
$$V_{\pm} = \frac{1}{2}(z^2 - 1)^2 \pm gz$$

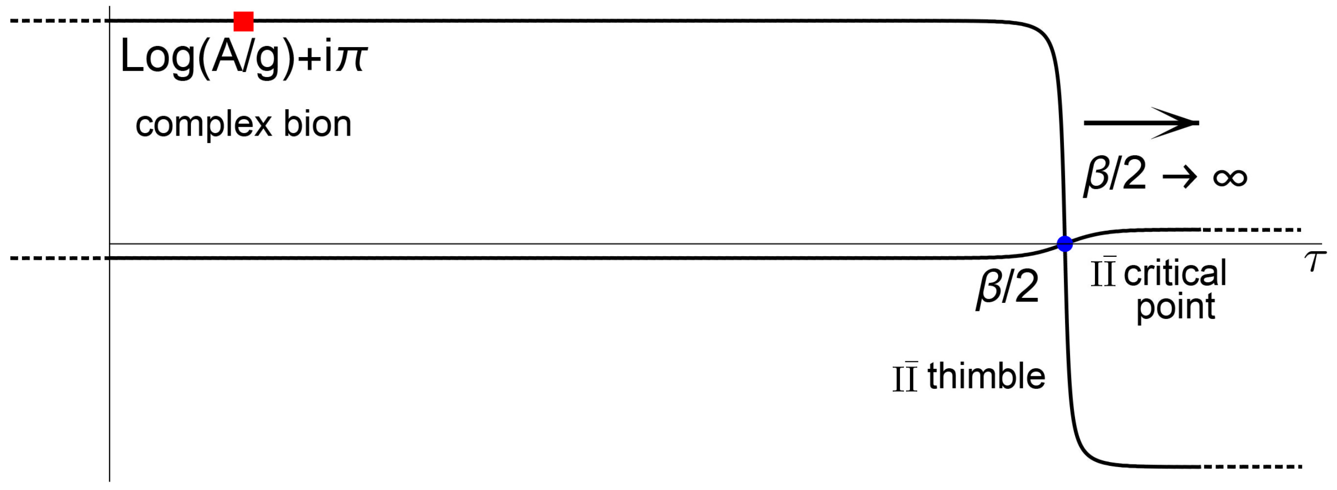
If complex bion is not included, we would conclude Susy is unbroken. Contradiction!

Exact bounce



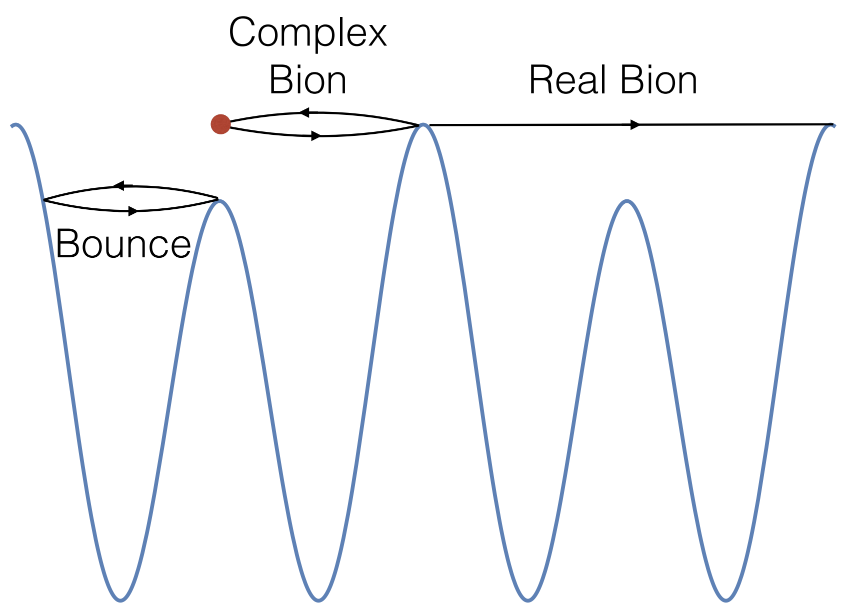
Exact complex bion solution





Periodic potential, real and complex bions

This system has Witten index zero but susy is known to be unbroken. Two ground states, Bose-Fermi paired.

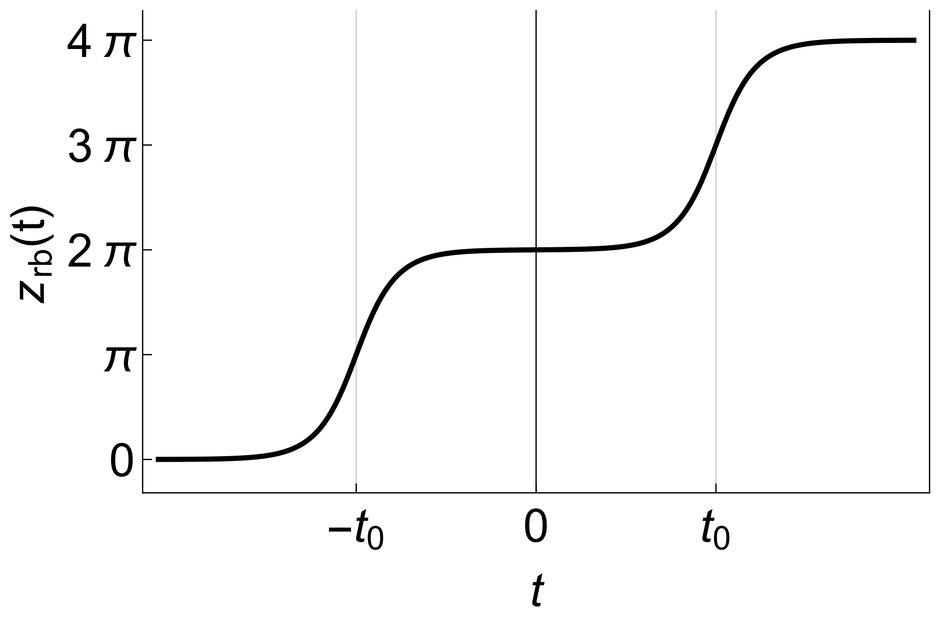


Inverted potential

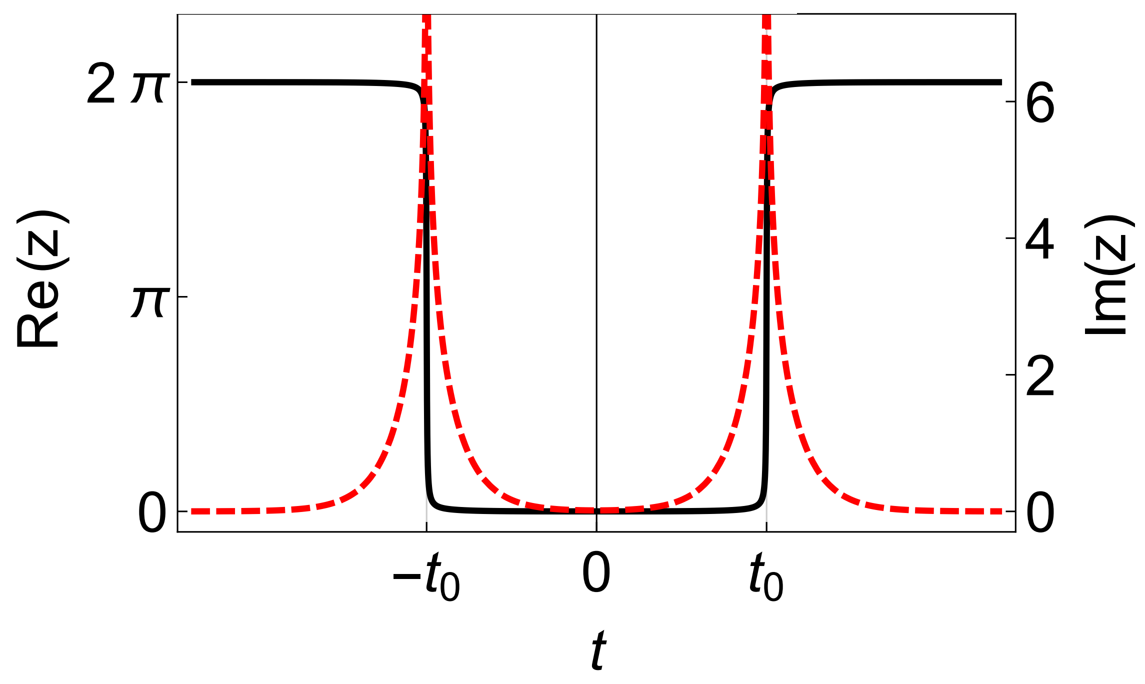
⇒If complex bion is not included, real bion renders ground state energy negative.
In violation of Susy algebra. (a would-be genuine disaster)!

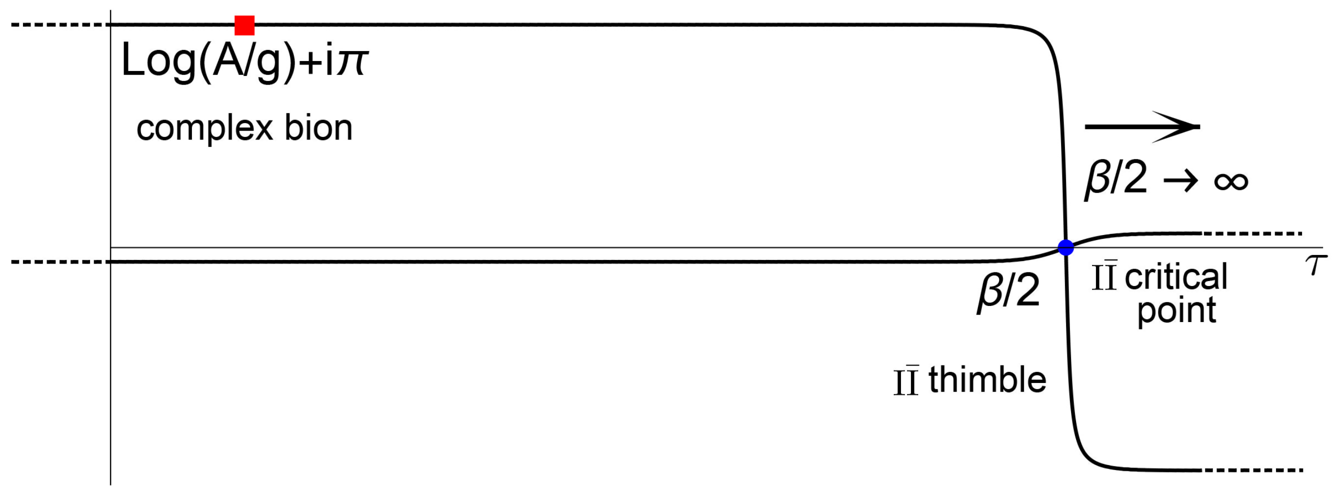
⇒Complex bion is strictly necessary. But it is not only multi-valued, but also singular. Yet, its action is finite.
Imaginary part of action $i\pi$. This is the hidden topological angle (HTA) (Behtash et.al.2015)
This is the sense in which we have to go through a change of perspective in path integrals! These are legit configurations contributing to path integral. (These are in Big "sins" category, according to ancient texts.)

Real bion



Exact complex bion





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Part II

Coupling TQFT to QM

Motivation: We learned a lot from $R_3 \times S_1$, but

0) Adiabatic continuity (strong coupling NP phenomena can be continuously connected to weak coupling NP phenomena). A conjecture, for which there is ample evidence.

1)Mechanism of mass gap generation in deformed YM, QCD(adj), and deformed QCD in any rep. ferm. Some very exotic mechanism, so much so that we could not guess them without solving, but rigorous in the weak coupling domain.

2)Absence of mass gap in chiral limit of QCD, derivation of chiral Lag.

3)Confinement in YM and QCD with fermions in rep R.

4)Mechanism of both discrete and continuous chiral symmetry breaking in QCD-like theories

5) Correct theta angle dependence, topological susceptibility

6) Understanding of semi-classical approach more deeply eventually lead to **"Resurgence in QFT and QM program".**

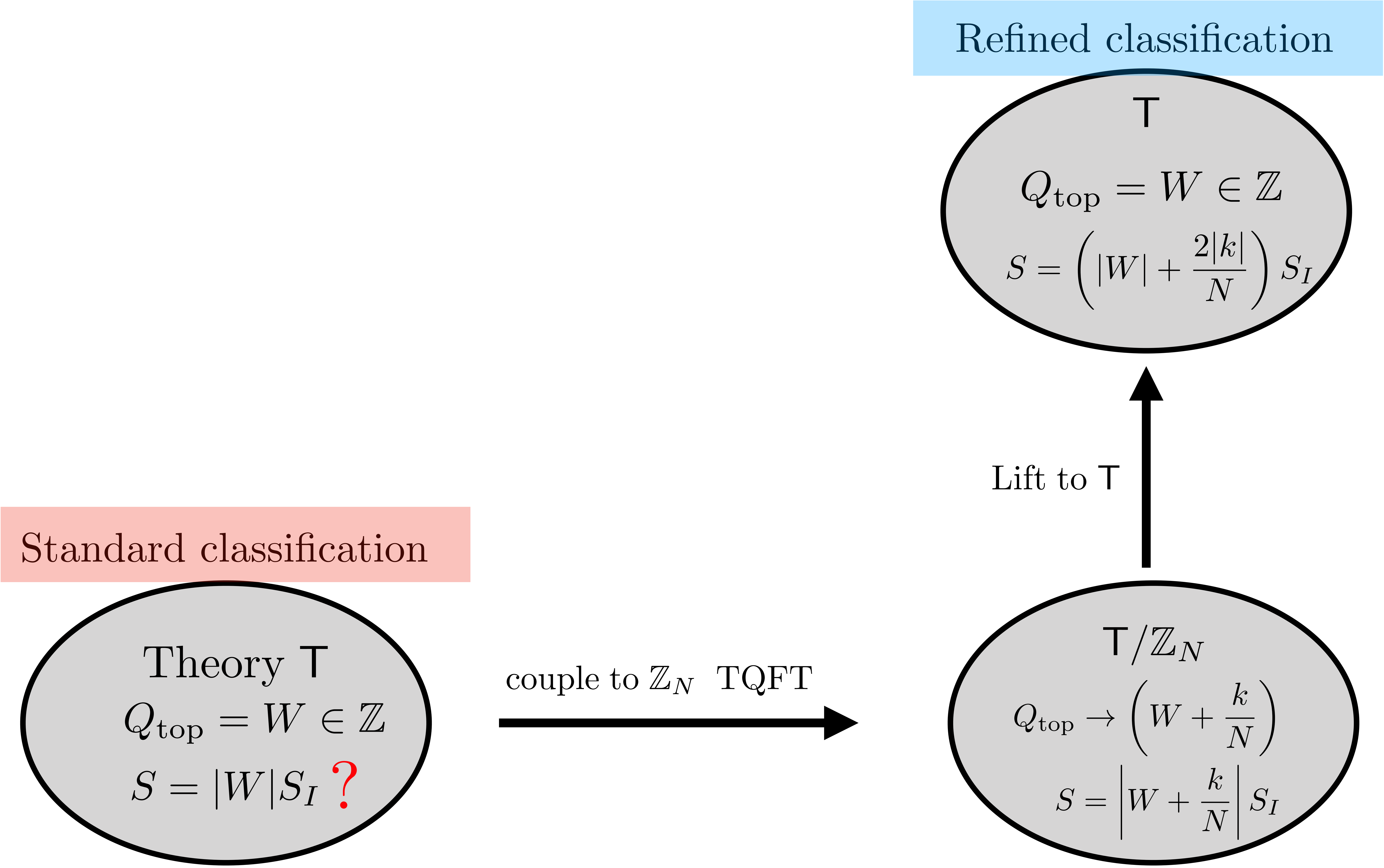
But I am quite disturbed by the following:

If weak coupling EFT on the calculable regime adiabatically connected to strong coupling regime knows so much about the strong coupling domain:

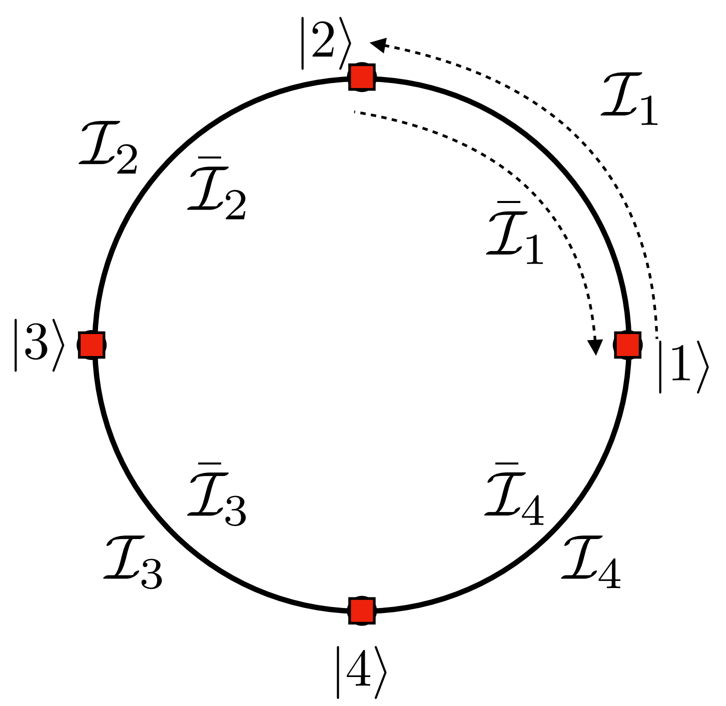
1) Doesn't some facts concerning very rich non-perturbative microscopic effects/dynamics/saddles of the weak coupling constructions on small $S_1 \times R_3$ survive in the strong coupling ?

2) Why can't we start studying strongly coupled dynamics on R_4 or arbitrarily large M_4 or R_d directly for d-dim QFT?

Main and surprising result of coupling TQFT to QFT



QM of particle on a circle with N-minima:
T_N vs $(T_N/Z_N)_p$ models

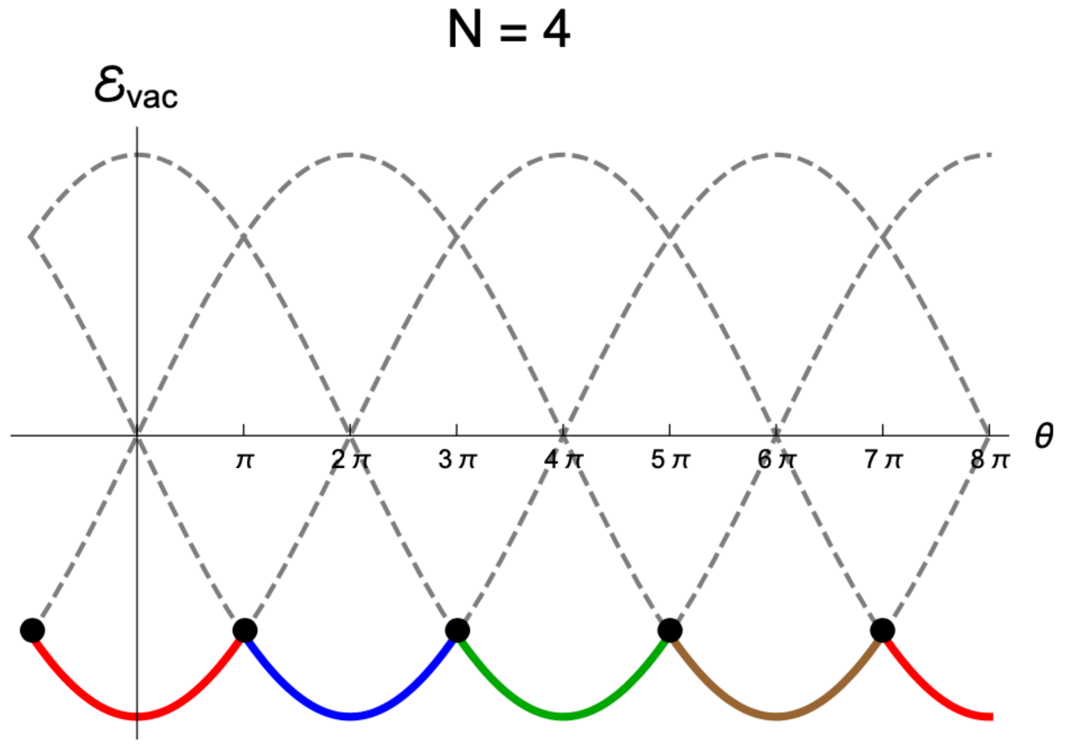


$$V(q) = -\cos(Nq), \qquad q \sim q + 2\pi$$

$$\mathbb{Z}_N : q \mapsto q + \frac{2\pi}{N}$$

theta term : $\mathrm{i}\frac{\theta}{2\pi}\int \mathrm{d}q$

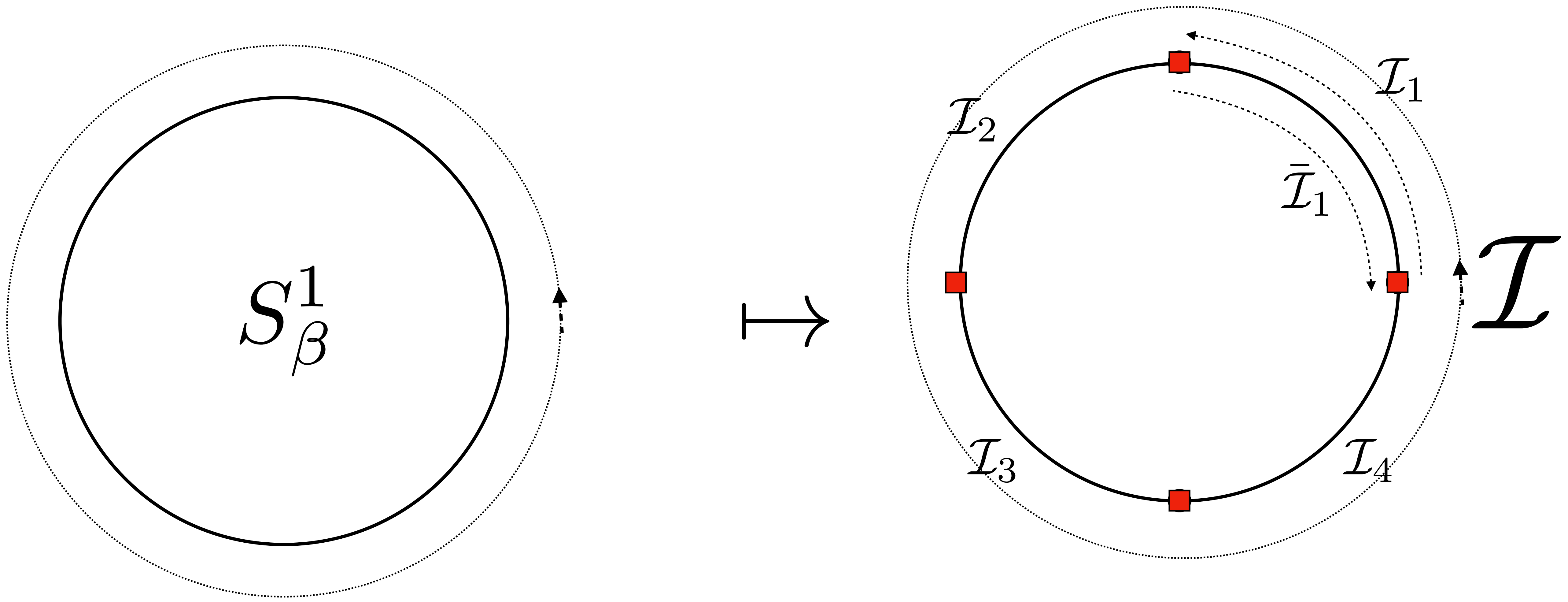
$$E_k(\theta) = -2K\mathrm{e}^{-S/N}\cos\frac{\theta + 2\pi k}{N}$$



In Born-Oppenheimer approximation,
we can work with tight-binding Hamiltonian,
and deal with only lowest N state,
which are split by NP effects.

Euclidean path integral

$$Z(\beta) = \mathrm{tr}[\mathrm{e}^{-\beta H}] = \int_{q(\beta)=q(0)} \mathcal{D}q \ \exp(-S[q]),$$



Partion function in Euclidean path integral: Sum over periodic paths, with integer topological charge.

$q(\tau) : S^1_\beta \to S^1, \ \ \pi_1(S^1) = \mathbb{Z}$ $\qquad$ $W = \frac{1}{2\pi} \int \mathrm{d}q \in \mathbb{Z}.$

Reverse engineering instanton sum

$$Z(\beta,\theta) = \sum_{k=0}^{N-1} \mathrm{e}^{2\beta K \mathrm{e}^{-\frac{S}{N}} \cos\frac{\theta+2\pi k}{N}}$$
$$= \sum_{k=0}^{N-1} \sum_{n=0}^{\infty} \sum_{\overline{n}=0}^{\infty} \frac{1}{n!}\frac{1}{\overline{n}!} \left(\beta K \mathrm{e}^{-\frac{S}{N}+i\frac{\theta+2\pi k}{N}}\right)^{n} \left(\beta K \mathrm{e}^{-\frac{S}{N}-i\frac{\theta+2\pi k}{N}}\right)^{\overline{n}}$$
$$= N \sum_{W\in\mathbb{Z}} \sum_{n=0}^{\infty} \sum_{\overline{n}=0}^{\infty} \frac{1}{n!}\frac{1}{\overline{n}!} \left(\beta K \mathrm{e}^{-S/N+\mathrm{i}\theta/N}\right)^{n} \left(\beta K \mathrm{e}^{-S/N-\mathrm{i}\theta/N}\right)^{\overline{n}} \delta_{n-\overline{n}-WN,0}$$

$$\sum_{k=0}^{N-1} \mathrm{e}^{\mathrm{i}2\pi k(n-\overline{n})/N} = N \sum_{W\in\mathbb{Z}} \delta_{n-\overline{n}-WN}$$

$$n-\overline{n}-WN=0, \qquad \text{i.e.,} \;\; n-\overline{n}=0 \mod N$$

More precisely: $n_1-\overline{n}_1 = n_2-\overline{n}_2 = \ldots = n_N-\overline{n}_N = W$

Contributing terms in the sum:

$$\mathrm{e}^{-\frac{S_I}{N}(n+\overline{n})}\;\mathrm{e}^{\mathrm{i}\frac{\theta}{N}(n-\overline{n})} = \mathrm{e}^{-\left(W+\frac{2\overline{n}}{N}\right)S_I}\;\mathrm{e}^{\mathrm{i}W\theta}$$

Integer topological charge, fractional action!

TQFT coupling to QM: Something sophisticated for something simple

Below, I will describe how to couple a TQFT to QM. This will describe an abstract formalism for something embarrassingly simple in QM. At the end of next few pages, you may even think why we did this at all.

What I will do is: In T_N model with Z_N symmetry, I will describe steps to turn on a classical background for Z_N or gauge Z_N completely.

The point is: The abstract formalism will cary over verbatim to Yang-Mills theory, QCD(adj), and with slight changes to QCD(F) (any flavor), as well as many other interesting QFTs. And will reveal insights which are otherwise not obvious to see.

$\mathbb{Z}_N$ TQFT

$$Z_{\text{top},p} = \int \mathcal{D}A^{(1)}\mathcal{D}A^{(0)}\mathcal{D}F^{(0)}\ \mathrm{e}^{\mathrm{i}\int F^{(0)}\wedge(NA^{(1)}-\mathrm{d}A^{(0)})+\mathrm{i}p\int A^{(1)}}$$

A sophisticated way of writing $\delta_{p,0}$ mod N.

$(A^{(1)}, A^{(0)})$ pair describe a $\mathbb{Z}_N$ gauge field that can be turned on in quantum mechanical T_N model to probe saddles, in particular, to probe the fractional instantons.

$$A^{(1)} \mapsto A^{(1)} + \mathrm{d}\lambda^{(0)}, \qquad A^{(0)} \mapsto A^{(0)} + N\lambda^{(0)}, \qquad F^{(0)} \mapsto F^{(0)}$$

To couple a classical $\mathbb{Z}_N$ background field to the q-field: $q \mapsto q - \lambda^{(0)}$,

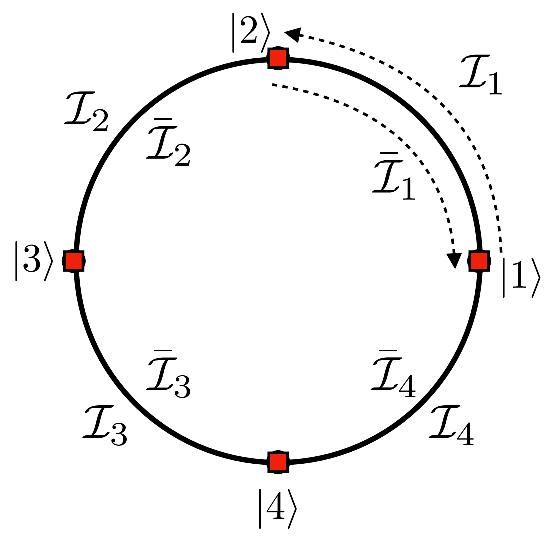
Gauge Inv. combos: $Nq + A^{(0)}$, $dq + A^{(1)} = (\dot{q} + A_\tau)\mathrm{d}\tau$

Kapustin, Seiberg, 2014

QM coupled to TQFT background

$$Z[(A^{(1)},A^{(0)}),p] = \int \mathcal{D}F^{(0)} \int_{q(\beta)=q(0)} \mathcal{D}q \, \mathrm{e}^{\mathrm{i}\int F^{(0)}\wedge(NA^{(1)}-\mathrm{d}A^{(0)})+\mathrm{i}p\int A^{(1)}}$$
$$\times \exp\left(-\frac{1}{g}\int d\tau\left(\tfrac{1}{2}(\dot{q}+A_\tau)^2-\cos(Nq+A^{(0)})\right)+\frac{\mathrm{i}\theta}{2\pi}\int(\mathrm{d}q+A^{(1)})\right)$$

Simple question: What does it calculate ?



Twisted BC = TQFT background

$$Z_\ell = \mathrm{tr}[\mathrm{e}^{-\beta H}\mathsf{U}^\ell] = \sum_{j=1}^{N}\langle j+\ell|\mathrm{e}^{-\beta H}|j\rangle = \int_{y(\beta)=y(0)+\frac{2\pi}{N}\ell}\mathcal{D}y\,\exp(-S[y]),$$

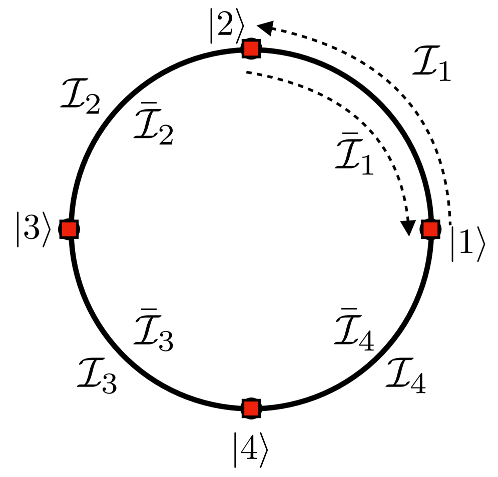
U: Translation operator, $\quad \ell = 0, 1, N-1$ fixed

One can trade TBC with Z_N background field. Use field redef.

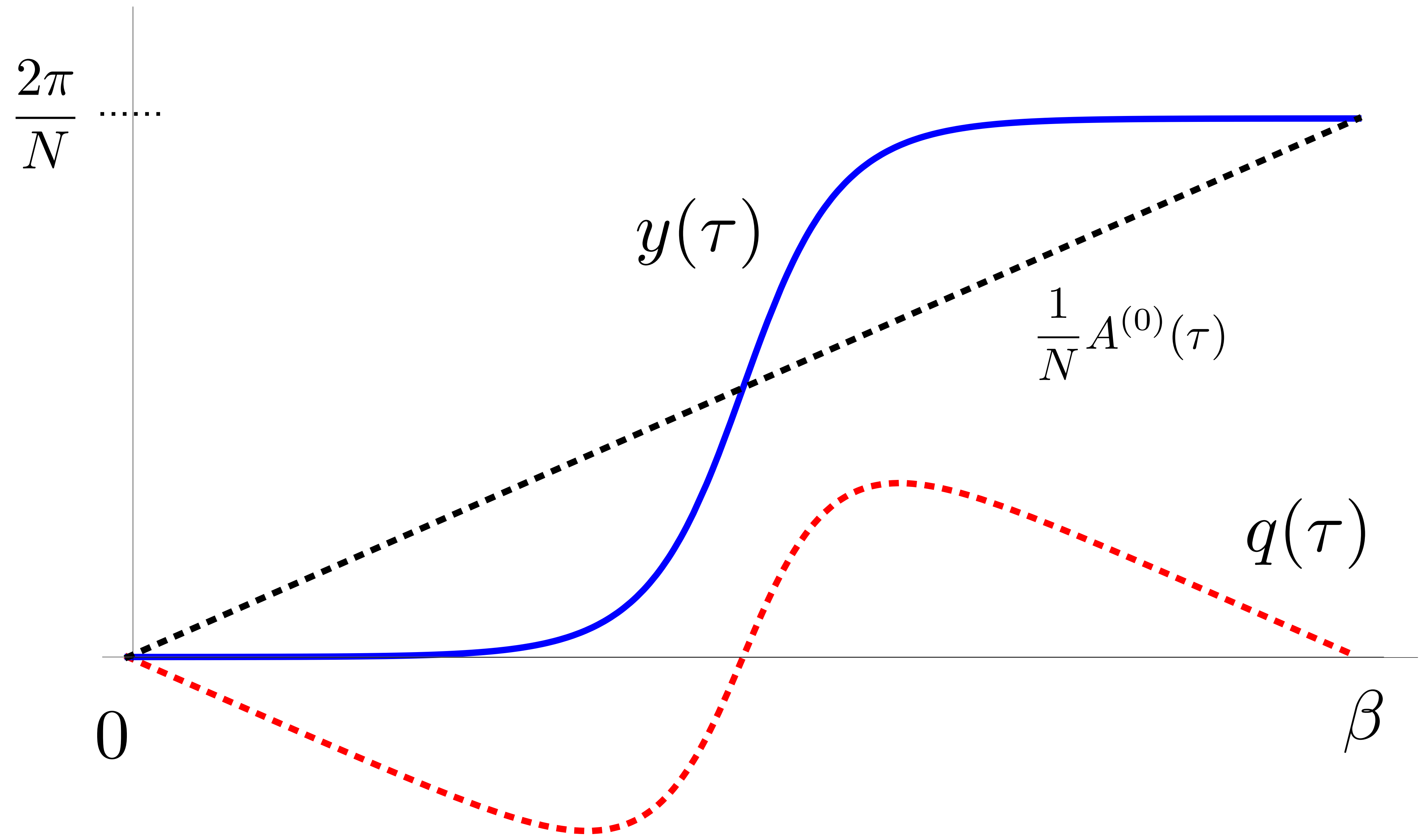
$$q(\tau) = y(\tau) - \frac{2\pi\ell}{N\beta}\tau, \qquad \text{hence } q(\beta) = q(0) \mod 2\pi.$$

$$S[q,\ell] = \frac{1}{g}\int d\tau\left[\frac{1}{2}\left(\dot q + \frac{2\pi\ell}{N\beta}\right)^2 - \cos\left(Nq + \frac{2\pi\ell}{\beta}\tau\right)\right] + \frac{\mathrm{i}\theta}{2\pi}\int\left(\mathrm{d}q + \frac{2\pi\ell}{N\beta}\mathrm{d}\tau\right)$$

which is nothing but Z_N TQFT coupled to QM.



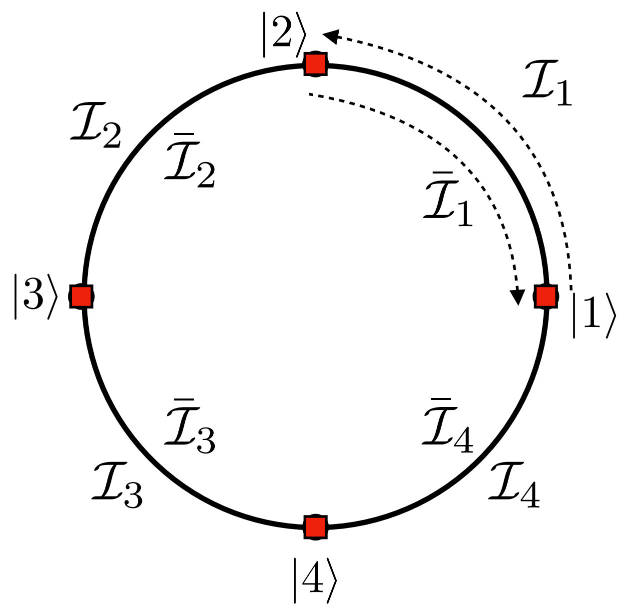
The instanton data (non-trivial topological charge) is transmuted to data about Z_N background gauge field.



TBC : $\quad y(\beta) = y(0) + \frac{2\pi}{N}\ell \qquad q(\tau) = y(\tau) - \frac{2\pi\ell}{N\beta}\tau,$

PBC : $\quad q(\beta) = q(0)$ + Z_N background gauge field

Gauging Z_N and $(T_N/Z_N)_p$ model



Gauging Z_N is equivalent to identifying adjacent sites.

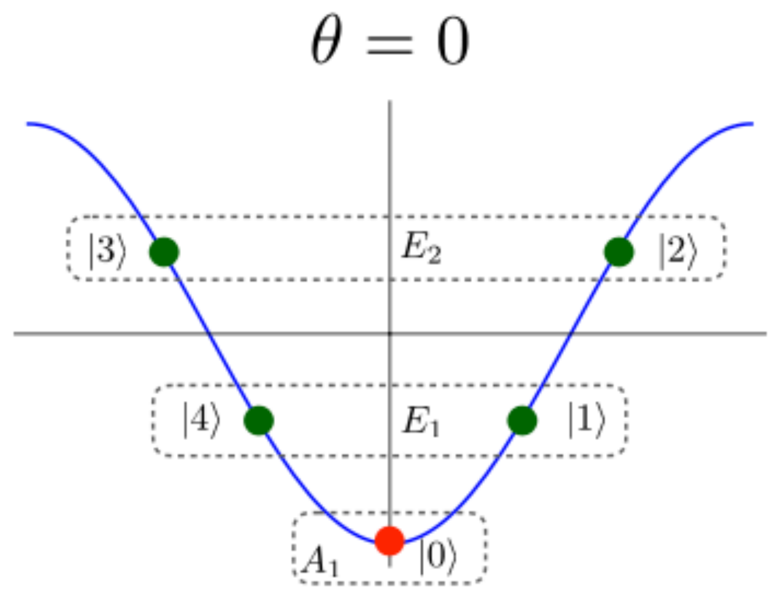
It dilutes Hilbert space by a factor of N.

N dimensional Hilbert space reduce to a 1 dimensional one.

$$Z_{(T_N/\mathbb{Z}_N)_p} = \int \mathcal{D}A^{(1)}\mathcal{D}A^{(0)}\ Z[(A^{(1)}, A^{(0)}), p]\ \delta(NA^{(1)} - \mathrm{d}A^{(0)})$$

$$\equiv \frac{1}{N}\sum_{\ell=0}^{N-1} \mathrm{e}^{-\mathrm{i}\frac{2\pi\ell p}{N}}\ Z_\ell$$

$$= \mathrm{e}^{\xi\cos\frac{\theta+2\pi p}{N}}$$



$$Z_{T_N} = \sum_{k=0}^{N-1} \mathrm{e}^{\xi\cos\frac{\theta+2\pi k}{N}}$$

Discrete theta angle θ_p = level p Chern-Simons = picking Bloch state with momentum p

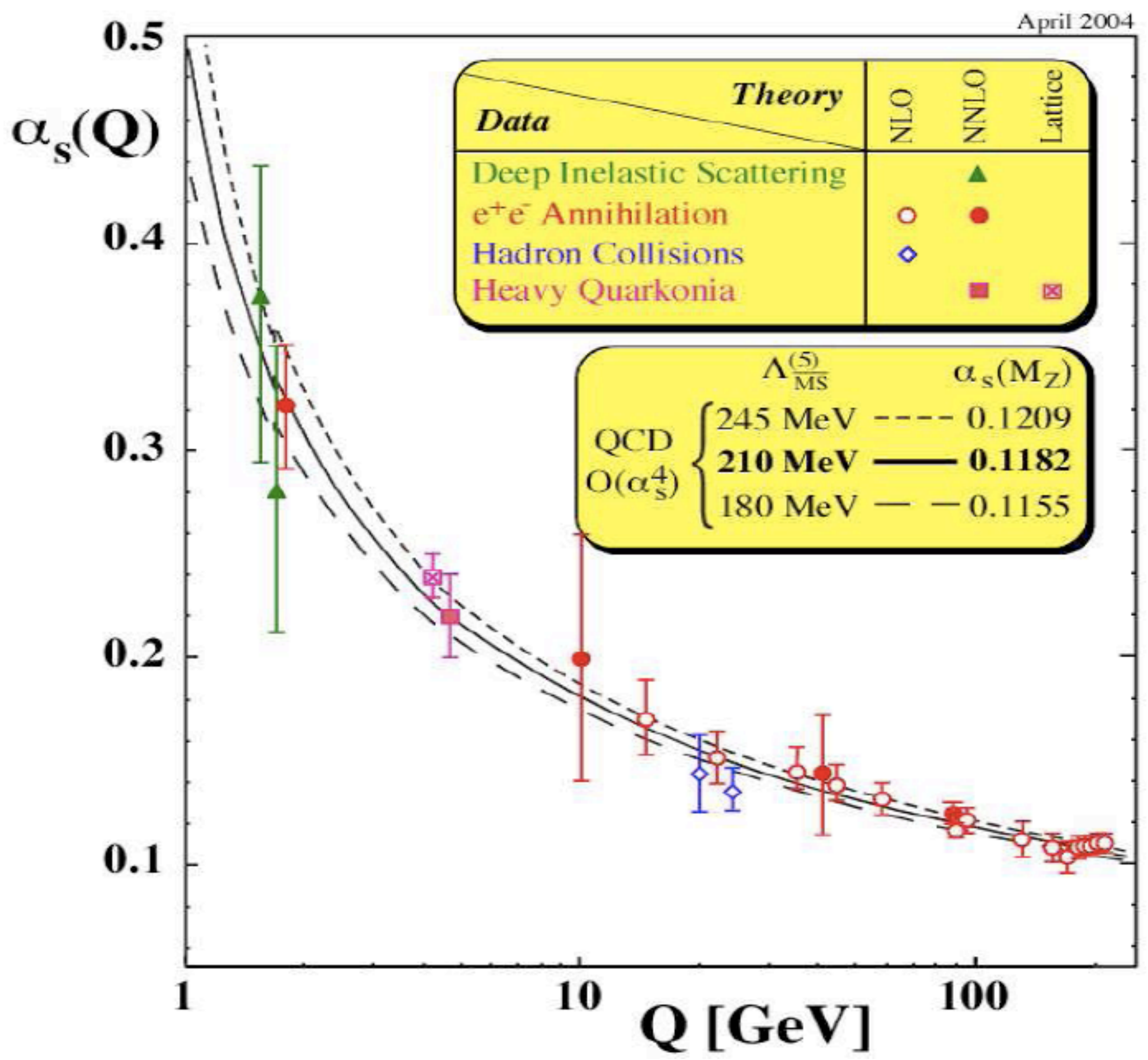
Part 3

Adiabatic continuity and Deformed Yang-Mills on

$$\mathbb{R}^3 \times S^1$$

Asymptotic Freedom

- asymptotic freedom
- Short distance: Weakly coupled, calculable...
- Long distance, strongly coupled. (Lattice works, analytical methods gloomy)



from: S. Bethke, hep-ex/0407021

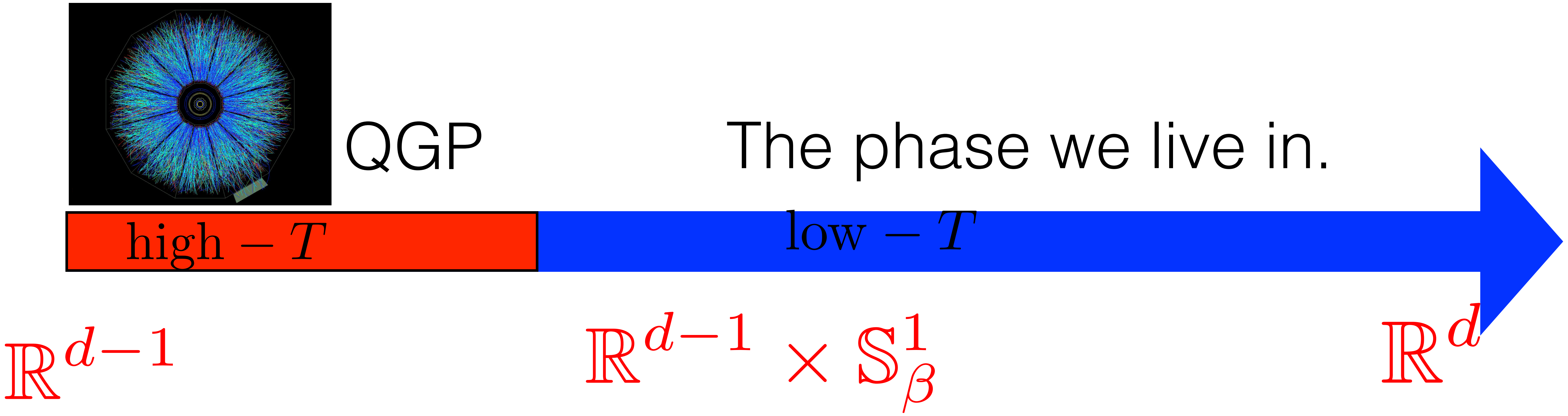
- Can we find a regime of asymptotically free gauge theories where the NP dynamics become calculable?

Adiabatic continuity and analyticity for YM?

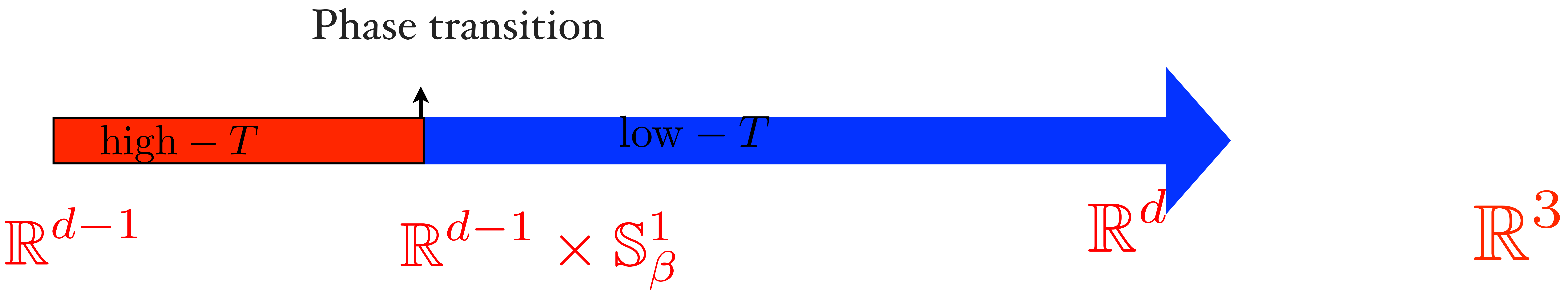
- We first want a (semi-classically) calculable regime of field theory, say of Yang-Mills or QCD. Of course, this is desirable. **But is it possible?**

- It is NOT known if such a framework exits on R4. In fact, theory becomes strongly coupled at longer distances for QCD-like theories.

- Consider these theories on four manifold R3 x S1, and study their dynamics as a function of radius. At small-radius, the theory is weakly coupled (thanks to asymptotic freedom) at the scale of the radius. But the theory is non-analytic as a function of radius, there is a phase transition.

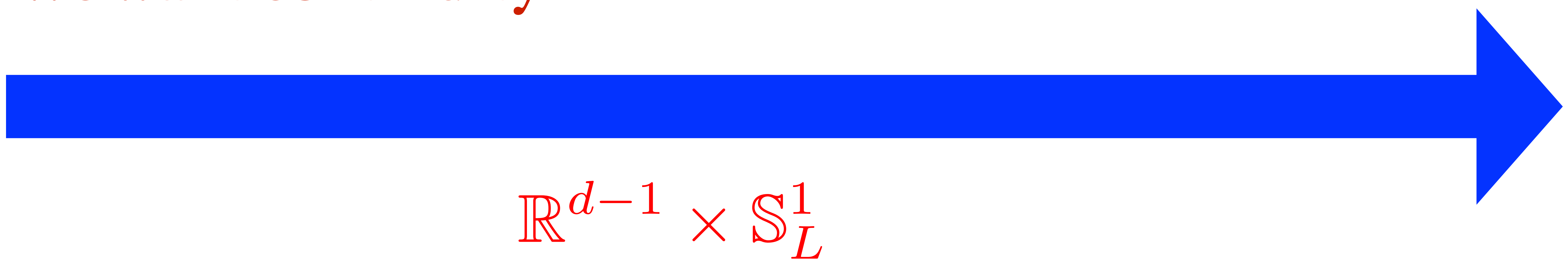


The idea of adiabatic continuity



Thermal: Rapid crossover/phase transition at strong scale

We want continuity



Adiabatic continuity and analyticity

Adiabatic continuity in non-susy theories is a spin-off of a brilliant idea by Eguchi and Kawai (82), called large-N reduction or volume independence.

What does EK say? It says something far more stronger than continuity, it implies volume independence, observable being independent of compactification radius at large-N. (Aleksey Cherman will talk about large-N.)

But it is tricky to achieve EK.

Large N volume independence or
"Eguchi-Kawai reduction" or "large-N reduction"

Theorem: $SU(N)$ gauge theory on toroidal compactifications of $\mathbb{R}^4$ to four-manifold $\mathbb{R}^{4-d} \times (S^1)^d$

No volume dependence in leading large N behavior of topologically trivial single-trace observables (or their connected correlators)

provided

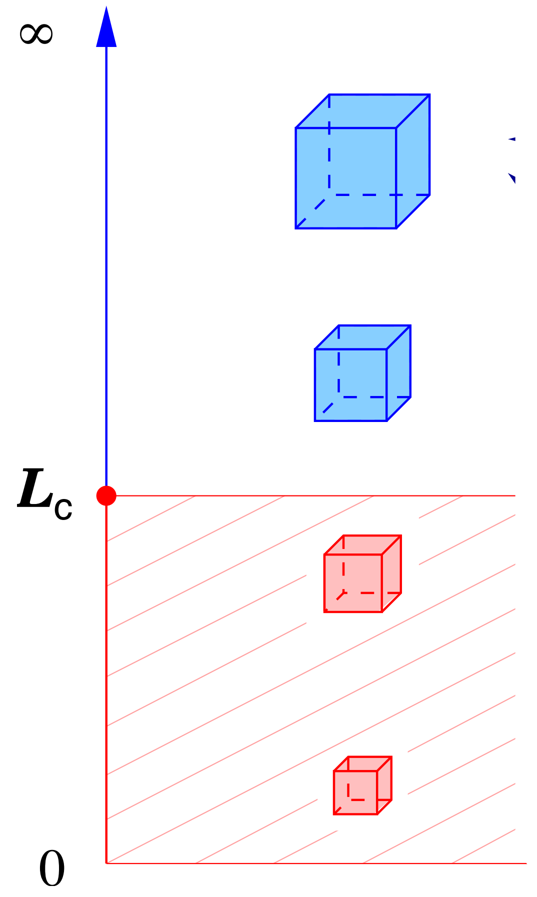
there are **no phase transitions as the volume of the space is shrunk.** More technically, **no spontaneous breaking of center symmetry** or translation invariance

Proof: Comparison of large N loop equaions (Eguchi-Kawai 82) in lattice gauge theory or \
N=∞ classical dynamics (Yaffe 82)

The **only** problem was that no-one was able to find **any** example of gauge theory in which **"provided"** holds. (and perhaps violating causality, an example already existed at the time EK was written. This is understood only 25 years later.)

Stumbling block

- Because of the attractiveness of the idea, much effort has been devoted. It was one of the hot subjects in mid-80's.
- However, there was always a phase transition when the space shrunk to small volume.
- Technically, an effective potential calculation in terms of Wilson lines (used to determine the phase of the small volume theory) gave a **negative** sign for **all** gauge theories. And we needed a positive sign! People gave up.

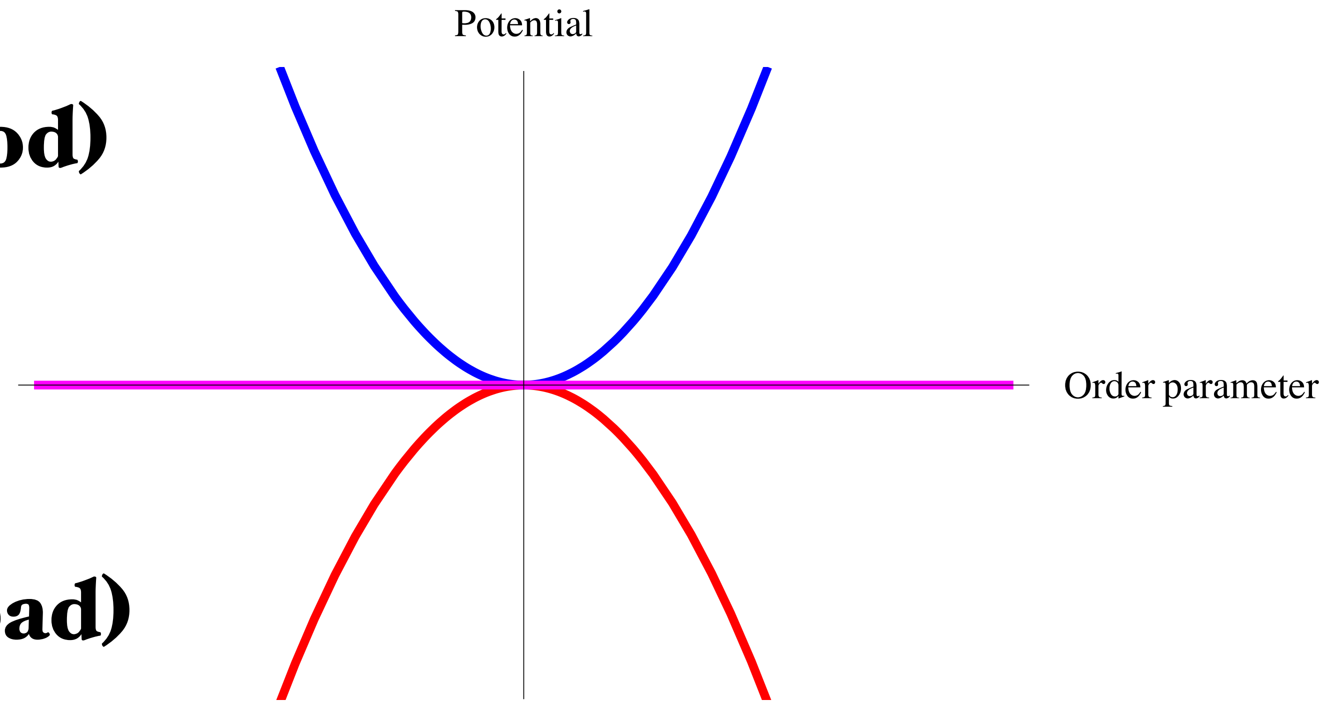


Instability

Stability (good)

Marginal

Instability (bad)



80's: EK, QEK, TEK.
Eguchi, Kawai, EK, Briilliantt, but fails

Gonzalez-Arroyo, Okawa, TEK, Failed, and REVIVED. (Many deep connections to non-commutative QFT, and recent works on TQFT coupling to QFT.)

Bhanot, Heller, Neuberger, QEK, Fails

Gross, Kitazawa, (YM Beta function from matrix model assuming working reduction. Clever.)

Yaffe,

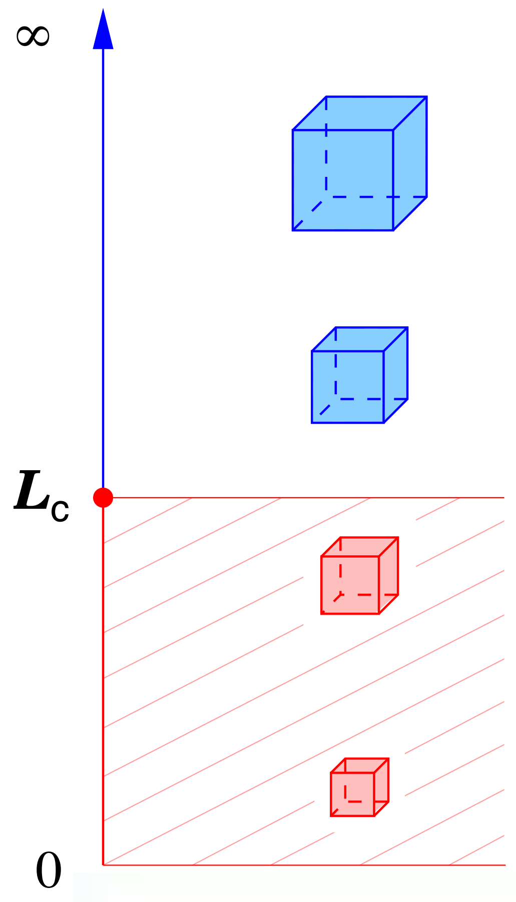
Migdal, Kazakov,
Parisi et.al.
Das, Wadia, Kogut,
+ 500 papers.... , but no single working example!

Stumbling block

- Because of the attractiveness of the idea, much effort has been devoted. It was one of the hot subjects in mid-80's.

- However, there was always a phase transition when the space shrunk to small volume.

- Technically, an effective potential calculation in terms of Wilson lines (used to determine the phase of the small volume theory) gave a **negative** sign for **all** gauge theories. And we needed a positive sign! People gave up.

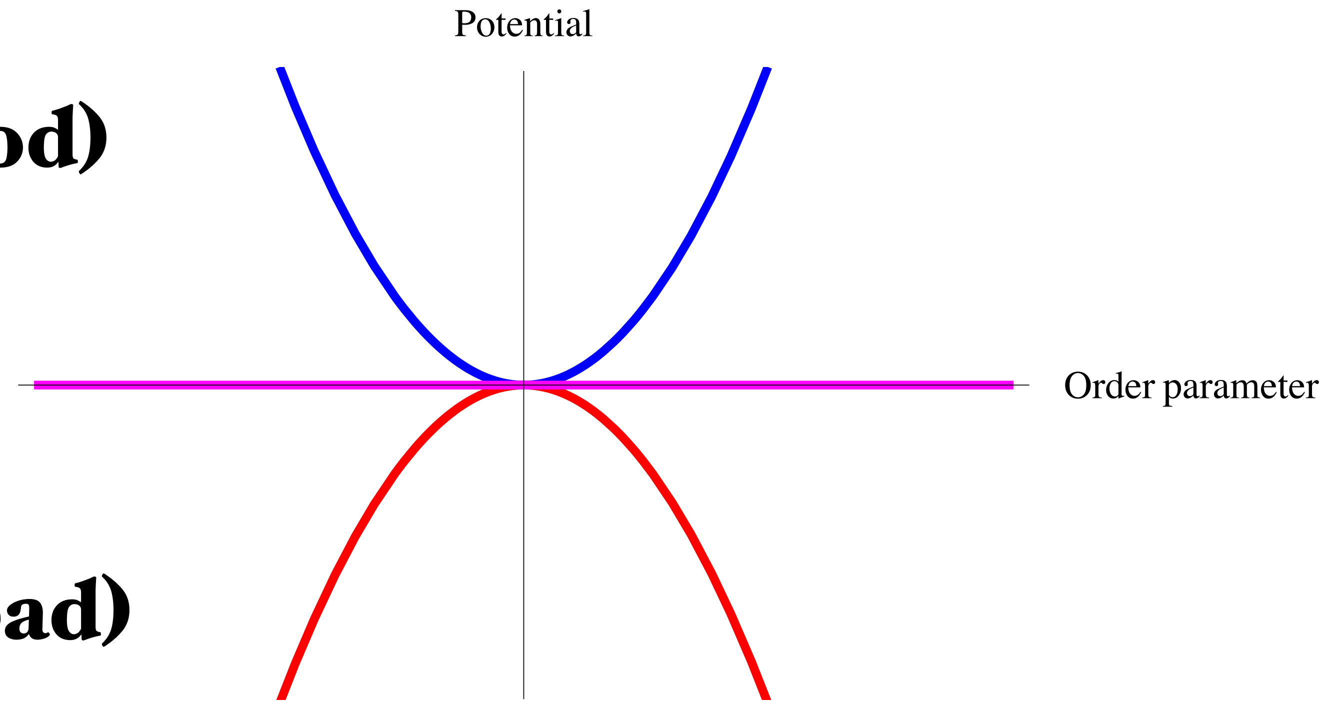


Instability

Stability (good)

Marginal

Instability (bad)



$\text{Yang} - \text{Mills on } \mathbb{R}^3 \times S^1$

- Z_N center symmetry, order parameter = Wilson line Ω

$$g(x+L) = hg(x), \qquad h^N = 1$$

Aperiodic gauge rotations, h $\in Z_N$

$$\text{tr}\Omega(x, x+L) \to h\ \text{tr}\Omega(x, x+L)$$

- $L > L_c$: unbroken center symmetry

$$\langle \text{tr}\ \Omega^n \rangle = 0$$

confined phase

- $L < L_c$: broken center symmetry

$$\langle \text{tr}\ \Omega^n \rangle \neq 0$$

deconfined plasma phase

failure of EK reduction

Gauge holonomy potential

$$V[\Omega] = -\frac{2}{\pi^2\beta^4}\sum_{n=1}^{\lfloor N/2 \rfloor}\frac{1}{n^4}\left|\mathrm{tr}\left(\Omega^n\right)\right|^2$$

Gross, Pisarski, Yaffe, 1981

Minimum at center-broken configuration. The value at min is the Stefan-Boltzmann law for gluons.

$$F = -\frac{\pi^2}{45}T^4(N^2-1)$$

At high-temperature YM theory, **this is inevitable and there is no room for negotiation.** This is also true in any QCD-like theory, and there is no hope here.

Evading the stumbling block(s)

In 2006, I realized that the analog of the effective potential calculation in supersymmetric gauge theory always gave zero.

But that requires using periodic boundary conditions for fermions. I was perfectly happy with it, and interpret it as **non-thermal compactification,** and realize that what you are calculating is not thermal partition function, but

$$\widetilde{Z}(L) = \mathrm{tr}[e^{-LH}(-1)^F]$$

What I did not know then: It was considered as another big "sin" to use periodic b.c. at least in a large-portion of non-supersymmetric QCD community.

At the heart of the super-symmetric cancelation was following identity:

Evading the stumbling block(s)

In 2006, I realized that the analog of the effective potential calculation in a supersymmetric gauge theory gave zero. At the heart of the cancelation was following identity:

$$-1 + 1 = 0$$

More precisely,

$$-1 \times (\text{stuff}) + 1 \times (\text{same stuff}) = 0$$

The crucial point: +1 appears due to the boundary conditions, and not supersymmetry!

Immediately, we deduce:

$$-1 + N_f > 0 \qquad \text{for } N_f > 1$$

Crucial Positive sign. (Hosotani did also show this in gauge-Higgs unification context, but its importance for confinement problem and large-N volume independence was not realized.) In QCD community, all earlier calculations were done for a specific (thermal) boundary condition.

Gauge holonomy potential QCD(adj) Nf-flavor

$$V[\Omega] = (N_f - 1)\frac{2}{\pi^2\beta^4}\sum_{n=1}^{\lfloor N/2 \rfloor}\frac{1}{n^4}\left|\mathrm{tr}\,(\Omega^n)\right|^2$$

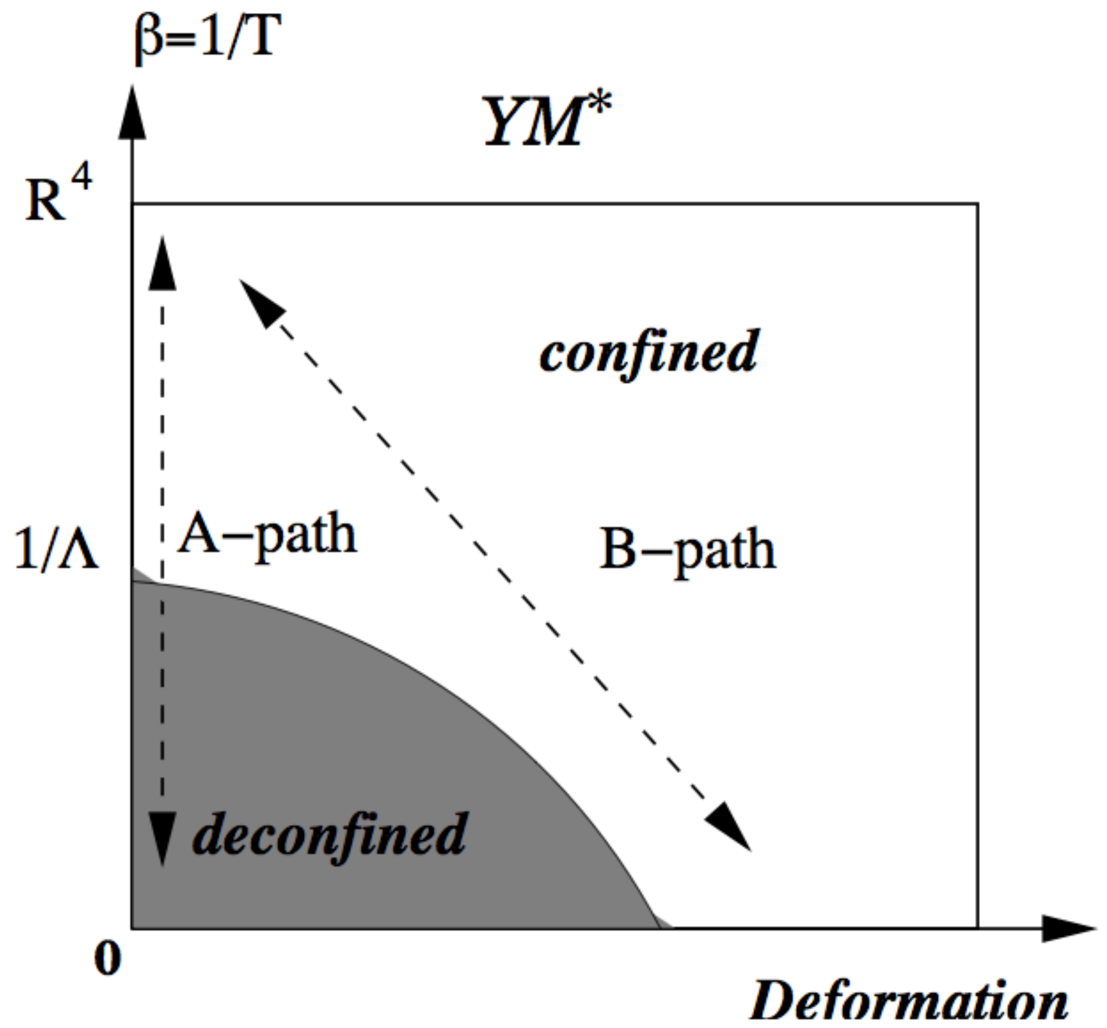
Kovtun, MU, Yaffe, 2007. Showed that QCD(adj) satisfies volume-independence, Eguchi-Kawai dream naturally.

This sign flip probably gave birth to one of the most promising windows to non-perturbative QCD. This is what I thought in 2007, and I will describe later in this talk. I believe it endures the test of time. And in the longer run, it is something that will remain.

Can we achieve center-stability in YM in small-L?

$$S^{\mathrm{YM}^*} = S^{\mathrm{YM}} + \int_{R^3\times S^1} P[\Omega(\mathbf{x})] \qquad\qquad P[\Omega] = A\frac{2}{\pi^2 L^4}\sum_{n=1}^{\lfloor N/2 \rfloor}\frac{1}{n^4}\left|\mathrm{tr}\,(\Omega^n)\right|^2$$

- Motivated by QCD(adj), Yaffe and I proposed a double-trace deformation that prevents center-breaking. (Yaffe, MU, 2008).



* We can now do reliable semi-classics here, and it is continuously connected to YM on R4.

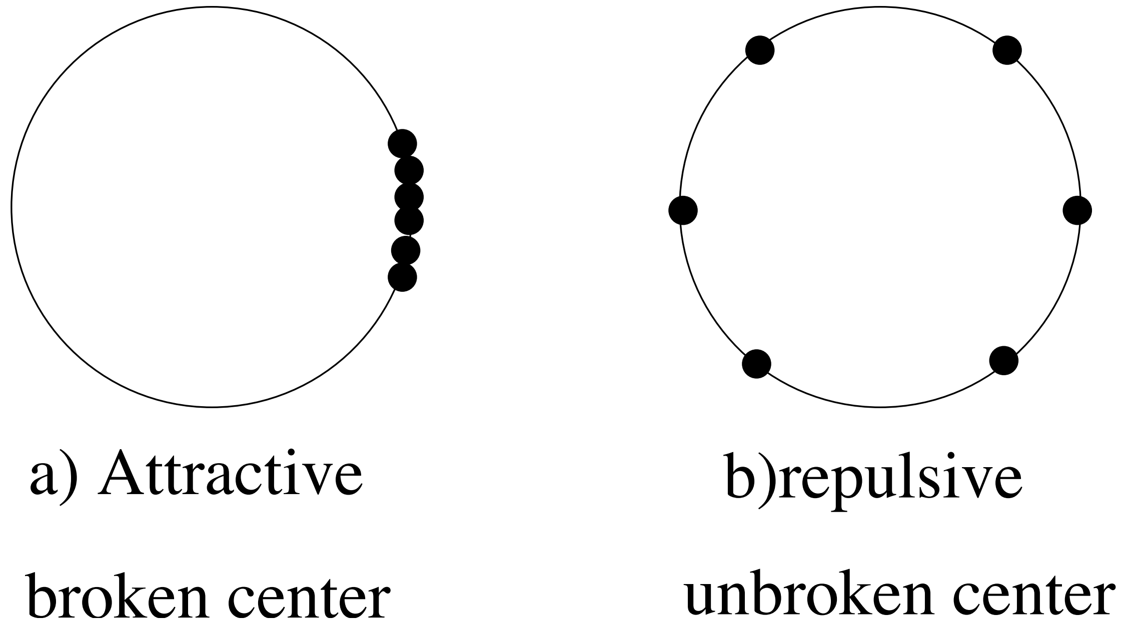
The double-trace deformation is something extremely interesting and has some very deep aspects especially in the context of large-N volume independence, but it is not my goal to discuss it in this talk.

Dimensional Reduction ? No, no!

- small L, asymptotic freedom, heavy, weakly coupled KK modes

- usual case: broken center symmetry
  $\langle \mathrm{tr}\,\Omega \rangle \neq 0 \Leftrightarrow$ eigenvalues clump

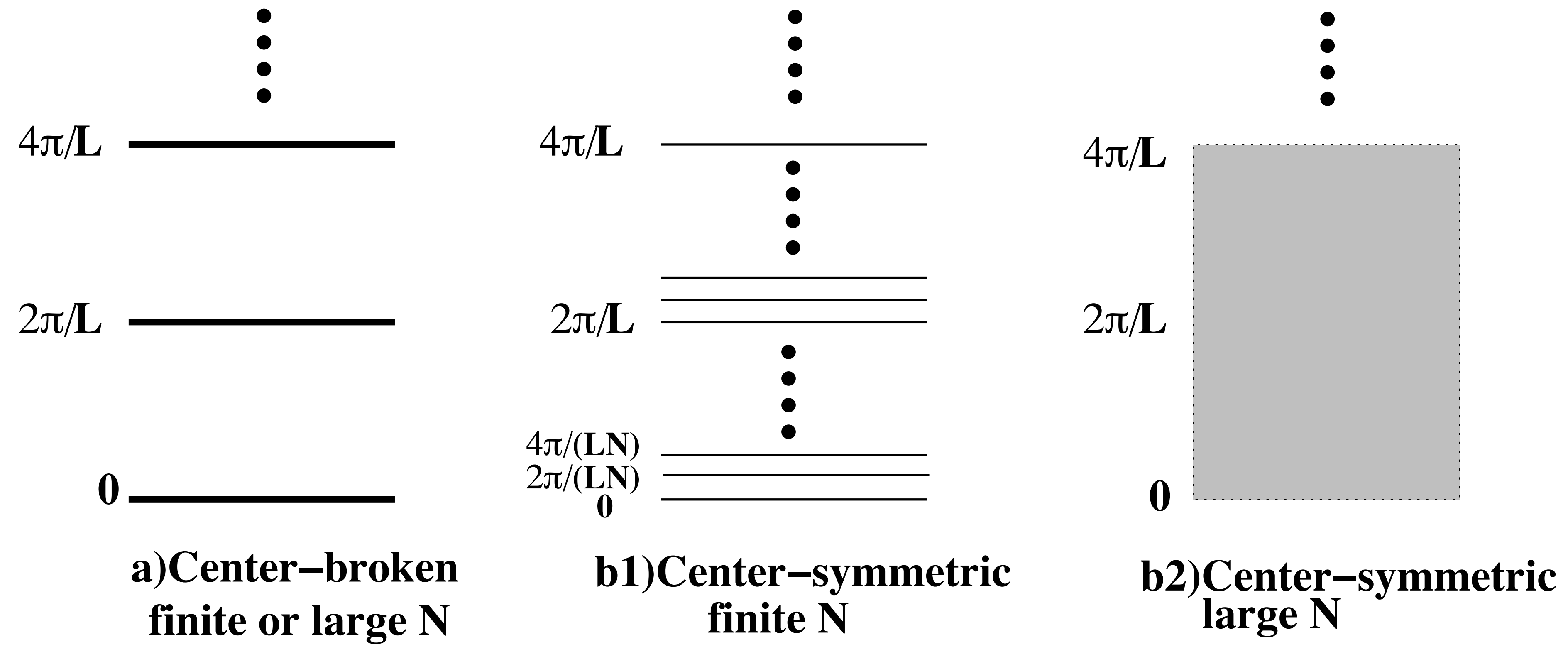
  $m_{KK} = 1/L,\ 2/L,\ \ldots,$
  perturbative control when $L\Lambda << 1$
  integrate out $\Rightarrow$ $3d$ effective theory, L-dependent



- center-symmetric case:
  $\langle \mathrm{tr}\,\Omega \rangle = 0 \Leftrightarrow$ eigenvalues repel

  $m_{KK} = 1/NL,\ 2/NL,\ \ldots,$
  perturbative control when $NL\Lambda << 1$

topological defects (instantons),
mass gap, confinement, later......



Topological configurations: Monopole-instantons

1-defects, Monopole-instantons: Associated with the N-nodes of the affine Dynkin diagram of SU(N) algebra.
The Nth type corresponds to the affine root and is present only because the theory is *locally 4d*!

van Baal, Kraan, (97/98),
Lee-Yi, Lee-Lu (97)

$$\mathcal{M}_k \sim e^{-S_k} e^{-\alpha_k \cdot b + i\alpha_k \sigma + i\theta/N}, \qquad k = 1, \ldots, N$$

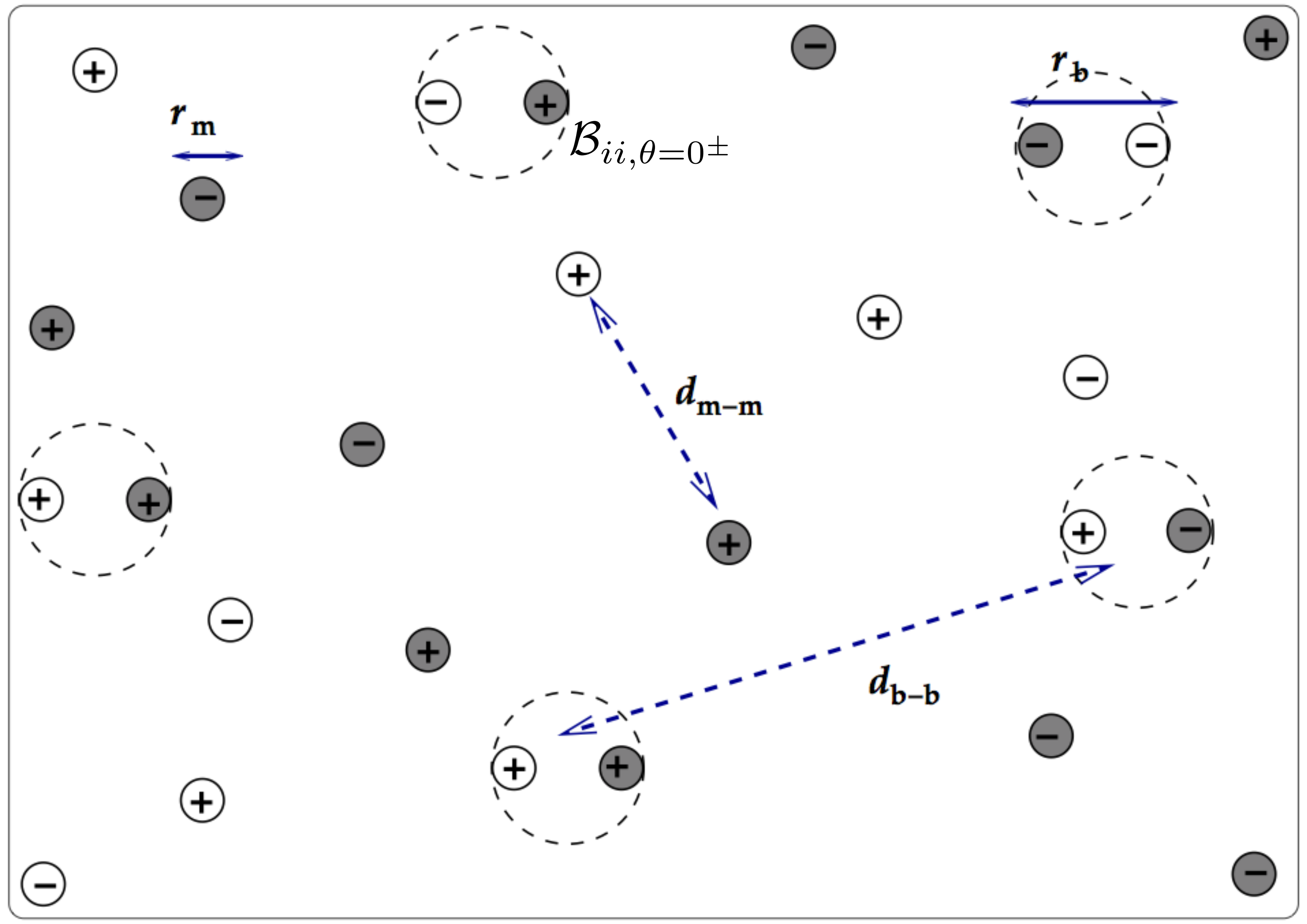
$$\boxed{S_k = \frac{8\pi^2}{g^2 N} = \frac{S_I}{N}}$$

Action 1/N of the 4d instanton, keep this in mind!

Proliferation of monopole-instantons generates a non-perturbative mass gap for gauge fluctuations, similar to 3d Polyakov model (Polyakov, 74).
It is first generalization thereof to local 4d theory!

Deformed YM, Euclidean vacuum

Dilute gas of monopole instantons and bion events



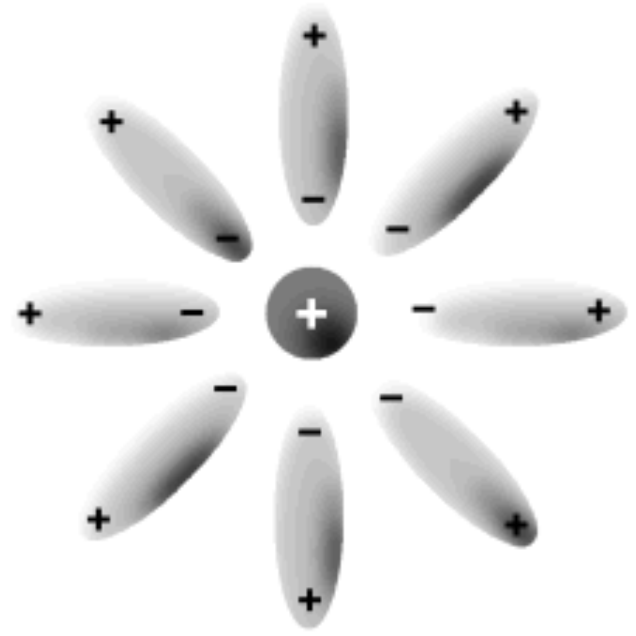
Relation to R4? Will comment on this later...

$$\langle F^2 \rangle_{0^\pm} \propto \mathcal{M}_i + [\mathcal{M}_i \bar{\mathcal{M}}_j] + [\mathcal{M}_i \bar{\mathcal{M}}_i]_{0^\pm} + \ldots$$

Ambiguity in condensate sourced by neutral bion.

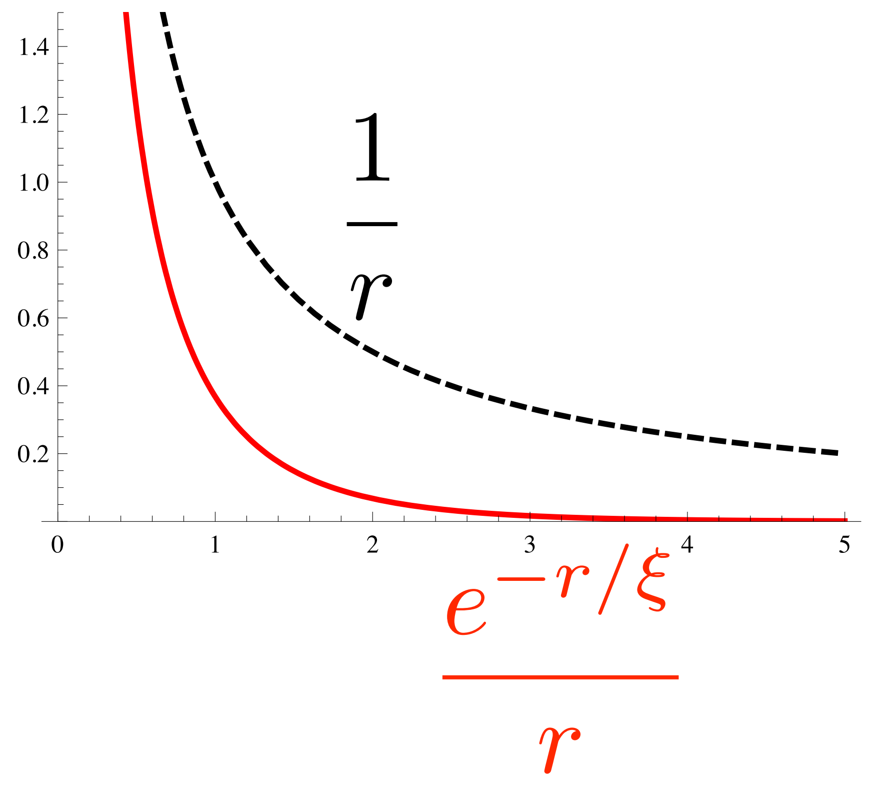
The essence of mass gap in Polyakov-mechanism in 3d

't Hooft-Polyakov monopole solutions (instantons in 3d) in Georgi-Glashow model. Polyakov 1977
Partition function of gauge theory = The grand canonical ensemble of classical monopole plasma. The field of external charge in a classical plasma decay exponentially. Debye-Hückel 1923. Proliferation of monopole-instantons generates mass gap for gauge fluctuations.



Due to screening

$$\frac{1}{r} \longrightarrow \frac{e^{-r/\xi}}{r}$$



Finite magnetic screening length=mass for gauge fluctuations for U(1) photon
= Confinement of electric charge (I will not show this part explicitly since I would like to emphasize mass gap. But the two are intimately related.)

Long-distance 3d dual theory

$$S^{\rm dual} = \int_{\mathbb{R}^3} \left[\underbrace{\frac{1}{2L}\left(\frac{g}{2\pi}\right)^2 (\nabla\sigma)^2}_{\text{Maxwell term}} - \zeta \sum_{i=1}^{N} \underbrace{\cos(\alpha_i\cdot\sigma)}_{\text{Monopole Operator}}\right].$$

Abelian duality

$$F^{(j)}_{\mu\nu} = \frac{g^2}{2\pi L}\,\epsilon_{\mu\nu\rho}\,\partial_\rho\sigma^j$$

monopole due to compactness of Higgs scalar

Lee, Yi, Kraan, vanBaal, 97, 98

Monopole charges

$$\Delta^0_{\rm aff} \equiv \{\underbrace{\alpha_1, \alpha_2, \ldots, \alpha_{N-1}}_{\text{usual N-1 monopoles}}, \overline{\alpha_N}\}\,.$$

Semi-classical Mass gap on R3 x S1

$$m_g^2 = \Lambda^2 (\Lambda L N)^{5/3} \max_k \cos \frac{\theta + 2\pi k}{N}$$

Mass gap monopole-instanton effect.

Expected non-trivial theta angle dependence (not present in Polyakov model).

For SU(2), mass gap vanishes at theta=pi. An exponentially smaller mass gap appears due to magnetic bion effects. The vacuum is 2-fold degenerate due to CP-breaking, as per magnetic bion induced potential.

Analysis strictly reliable for $(\Lambda L N) \lesssim 1$

Two remarkable result from lattice simulations of deformed theory at small S1 x R3: Topological susceptibility

Topological susceptibility in SU(4) dYM on small S1 x R3 vs Pure YM on the confined phase approximately R4. The deformation parameters for single winding and double winding loop is denoted by h.

Green curve is roughly the sharp drop associated with the deconfinement phase transition.

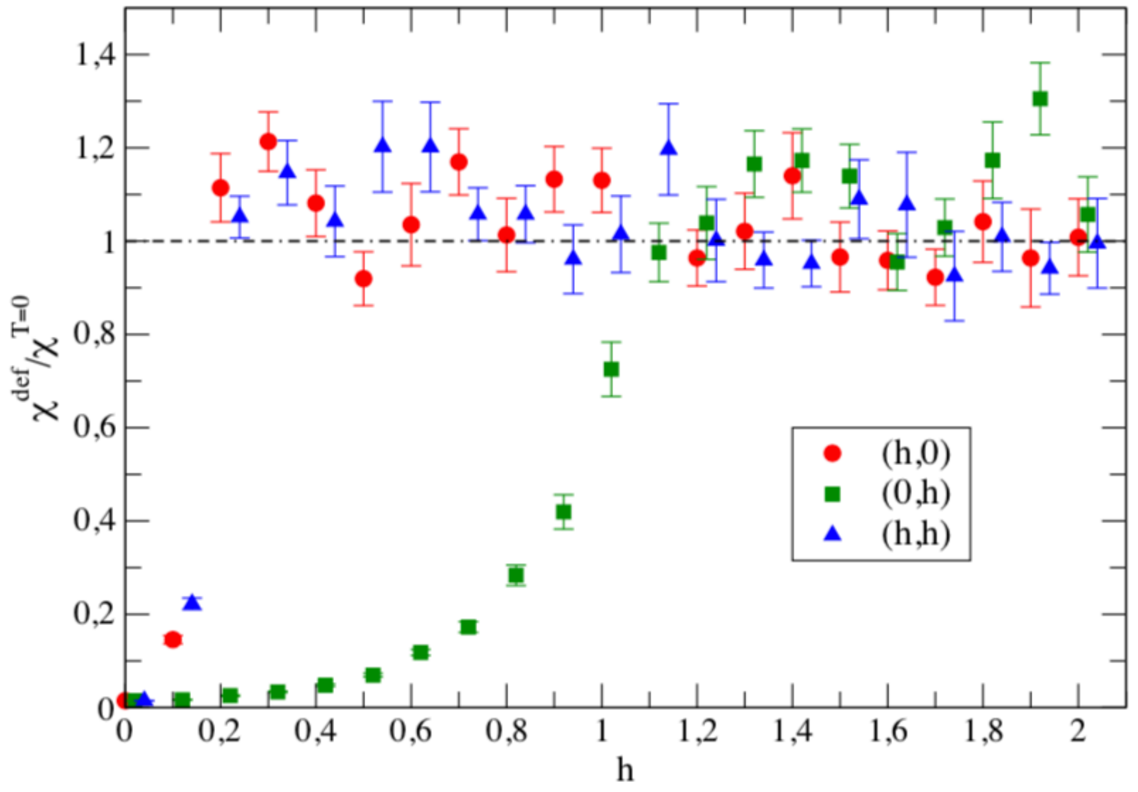


FIG. 11: Ratio between the topological susceptibility χ computed in the deformed theory and the one at $T = 0$ computed in Ref. [16] for different values of the deformation parameters h_1 and h_2. Results are obtained on the 6×32^3 lattice at bare coupling $\beta = 11.15$.

Bonati, Cardinali, D'Elia, Mazziotti, 2019

The simulation results strongly suggest us that we should carefully think about deformed YM. Clearly, it knows something deep about YM on R4!

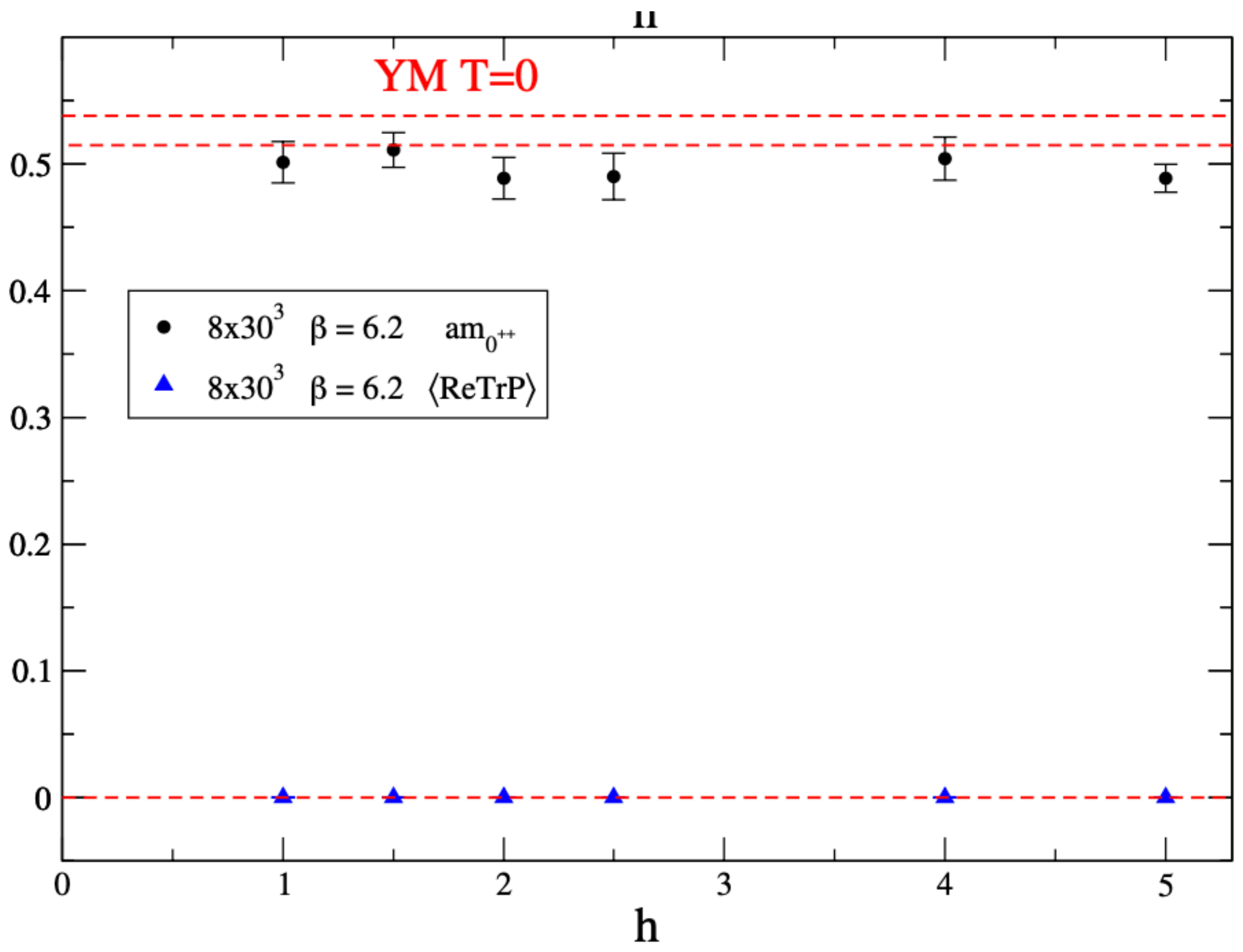
Two remarkable result from lattice simulations of deformed theory at small S1 x R3: Mass gap

Mass gap in SU(3) dYM on small S1 x R3 where circle size is roughly half of the strong length scale (500MeV). (Theory without deformation would be well in the deconfined phase.)

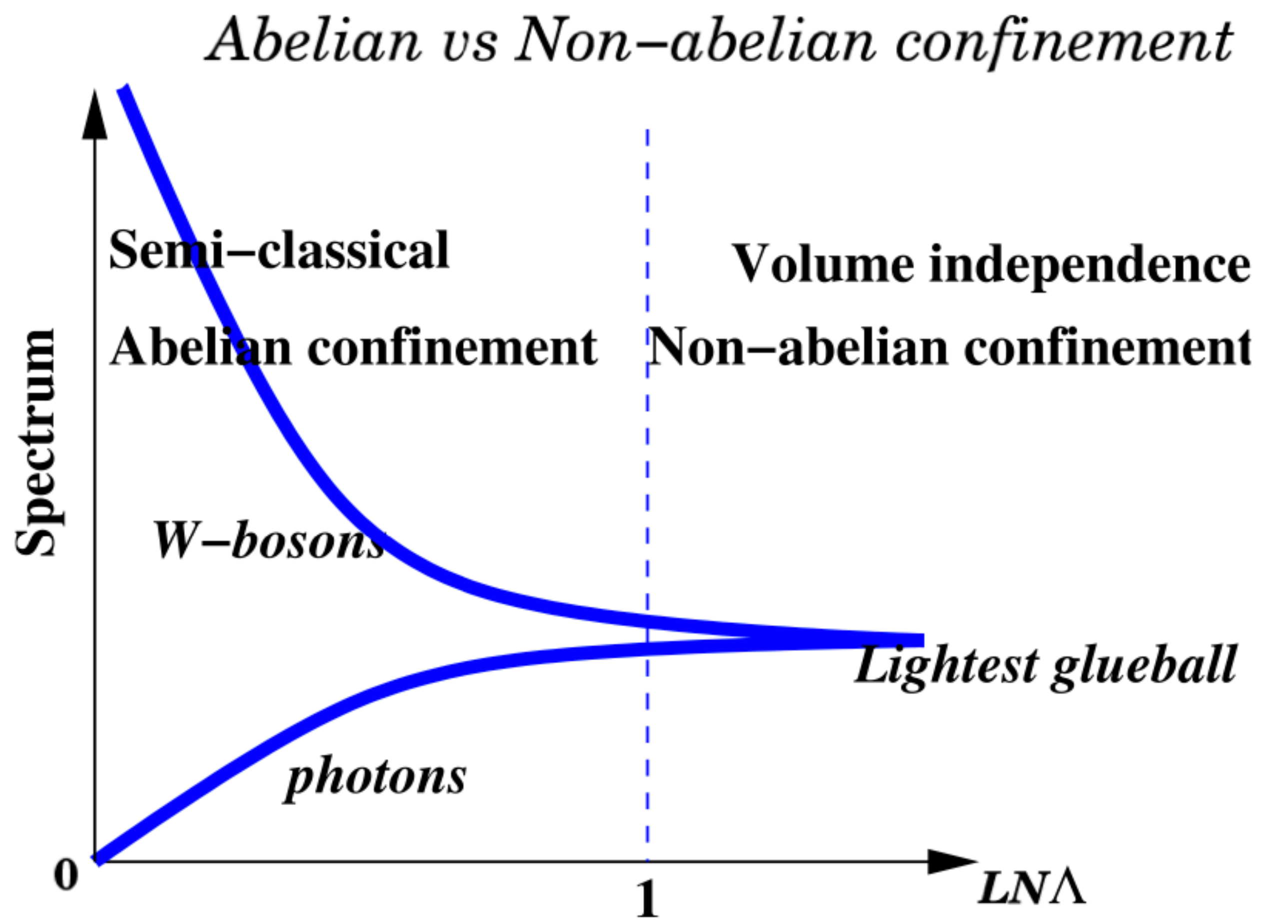
Pure YM on the zero temperature confined phase approximating R4.

The deformation parameters is denoted by h.

Remarkable agreement between small-circle deformed theory and zero temperature pure YM.
(in exact correspondence with observations in Yaffe/MU, and Shifman/MU (2008)



Athenodorou, Cardinali, D' Elia
To appear soon!



This results tell us to take dYM far more seriously and to think much harder.

Mini-space formalism: What are the NP configurations that contribute to partition function of dYM on R3 x S1?

mentioned in Tanizaki, MU “Modified instanton sum in QCD and higher-groups”, details in “Strongly coupled QFT dynamics via TQFT coupling “

The path integration over the fields $\int \mathcal{D}\boldsymbol{\sigma}\ \mathrm{e}^{-S}$ has a zero mode part. In this subspace, the measure reduce to an ordinary integral over the fundamental cell of $\boldsymbol{\sigma}$ field.

$$
\begin{aligned}
z(\theta) &= \int_{\text{cell}} [d\boldsymbol{\sigma}]\ \mathrm{e}^{-s} \\
&= \left(\prod_{i=1}^{N-1} \int_0^{2\pi} d\sigma_i \right) \prod_{a=1}^{N} \mathrm{e}^{\xi \mathrm{e}^{\mathrm{i}\left(\boldsymbol{\alpha}_a\cdot\boldsymbol{\sigma}+\frac{\theta}{N}\right)}} \mathrm{e}^{\xi \mathrm{e}^{-\mathrm{i}\left(\boldsymbol{\alpha}_a\cdot\boldsymbol{\sigma}+\frac{\theta}{N}\right)}} \\
&= \prod_{a=1}^{N} \left(\sum_{n_a=0}^{\infty} \sum_{\overline{n}_a=0}^{\infty} \right) \frac{\delta_{n_1-\overline{n}_1, n_N-\overline{n}_N} \ldots \delta_{n_{N-1}-\overline{n}_{N-1}, n_N-\overline{n}_N}}{n_1!\overline{n}_1! \ldots n_N!\overline{n}_N!} \\
&\quad \xi^{n_1+\ldots+n_N+\overline{n}_1+\ldots+\overline{n}_N} \mathrm{e}^{i\frac{\theta}{N}(n_1+\ldots+n_N-(\overline{n}_1+\ldots+\overline{n}_N))}
\end{aligned}
$$

$$n_1 - \overline{n}_1 = n_2 - \overline{n}_2 = \ldots = n_N - \overline{n}_N = W$$

Looks familiar? Same condition as in our QM T_N model, not an accident!
Magnetic neutrality guaranteed by zero-mode integration and it automatically enforces integer quantization of topological charge! See also Diakonov and Petrov.

What are we summing over in dYM on R3 x S1?

Configurations that contribute to the partition function possess integer topological charge, but fractional action!

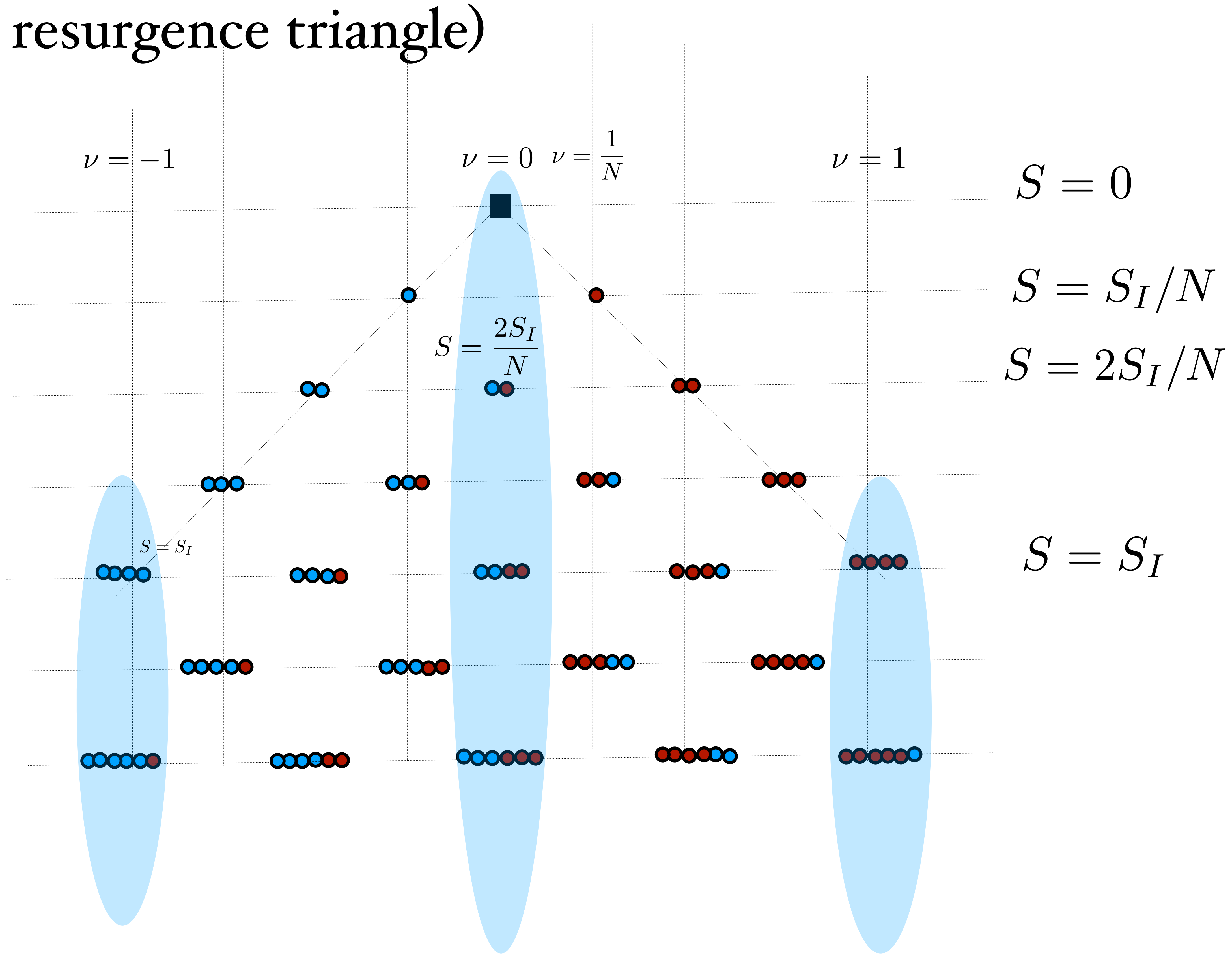
$$S = \frac{S_I}{N}(2\overline{n}_1 + \ldots + 2\overline{n}_N) + S_I|W| \in S_I\left(\frac{2}{N}|k| + |W|\right), \qquad k, W \in \mathbb{Z}$$

$$Q = W \in \mathbb{Z}$$

The sum is still over integer topological charge just like the BPST instanton on R_4 , but there is something intriguing going on about action. It does satisfy BPS bound, but exhibits a far more refined structure!

What are we summing over in dYM on R3 x S1?

The sum is still over integer topological charge just like the BPST instanton on R_4 , but there is something intriguing going on about action. It does satisfy BPS bound, but exhibits a far more refined structure! (graded resurgence triangle)



Part 4

TQFT coupling in Yang-Mills

Coupling Z_N TQFT to YM-formally

To turn on a classical background gauge field for the $\mathbb{Z}_N^{[1]}$ 1-form symmetry, introduce pair of $U(1)$ 2-form and 1-form gauge fields $(B^{(2)}, B^{(1)})$ satisfying

$$NB^{(2)} = \mathrm{d}B^{(1)}, \qquad N\int B^{(2)} = \int \mathrm{d}B^{(1)} = 2\pi\mathbb{Z}$$

Promote $SU(N)$ gauge field to a $U(N)$: $\widetilde{a} = a + \frac{1}{N}B^{(1)}$

Kapustin, Seiberg, 2014,
Komargodski et.al. 2017

$1-$ form gauge trans. and coupling TQFT :

$$B^{(2)} \mapsto B^{(2)} + \mathrm{d}\Lambda^{(1)}, \qquad B^{(1)} \mapsto B^{(1)} + N\Lambda^{(1)}$$

$$\widetilde{a} \mapsto \widetilde{a} + \Lambda^{(1)}, \qquad \widetilde{F} \mapsto \widetilde{F} + \mathrm{d}\Lambda^{(1)}$$

$$S[B^{(2)}, B^{(1)}, \widetilde{a}] = \frac{1}{2g_{\mathrm{YM}}^2}\int \mathrm{tr}[(\widetilde{F} - B^{(2)}) \wedge \star(\widetilde{F} - B^{(2)})] + \frac{\mathrm{i}\,\theta_{\mathrm{YM}}}{8\pi^2}\int \mathrm{tr}[(\widetilde{F} - B^{(2)}) \wedge (\widetilde{F} - B^{(2)})]$$

Modified instanton equation: $(\widetilde{F} - B^{(2)}) = \mp \star (\widetilde{F} - B^{(2)})$

Action: $$S = \mp \frac{8\pi^2}{g^2} \frac{1}{8\pi^2} \int \mathrm{tr}[(\widetilde{F} - B^{(2)}) \wedge (\widetilde{F} - B^{(2)})] = \frac{S_I}{N}$$

because $$\frac{N}{8\pi^2} \int B^{(2)} \wedge B^{(2)} \in \frac{1}{N}\mathbb{Z}$$

In SU(N) theory coupled to Z_N background gauge field, the configurations which satisfy BPS bound have action S_I/N, just like our monopole-instantons on R_3 x S_I. Is this an accident? Are they related?

I will make above very formal stuff first a bit more concrete (twisted BC a la 't Hooft), and then, even more concrete, describe in Hamiltonian formalism, in my own terms.

TQFT coupling= 't Hooft TBC

Not surprisingly, TQFT background can be traded with 't Hooft twisted boundary conditions.

't Hooft (1981) found constant topological charge 1/N and action 1/N configurations for certain aspect-ratio of T4. (He mentions that the reason for writing the article about a constant solution was the difficulty in finding them.) Historically, however, it was not easy to determine the time or space-time dependent non-trivial solutions.

Gonzalez-Arroyo, Garcia Perez et.al. (1990s-) found by numerical lattice simulations on latticized T3 x R that time-dependent fractional instanton solutions with action 1/N exist in the presence of 't Hooft flux.

I would like to argue that monopole-instantons are non-trivial configurations in the PSU(N) bundle! These are called 't Hooft-Polyakov monopoles, but 't Hooft did not realize or even come close to understanding their role in PSU(N) bundle despite the fact that he searched for non-trivial configurations in PSU(N) bundle. Understanding this requires many things that happened after 2008 papers I wrote with Yaffe and Shifman, and interesting work by Cherman and Poppitz 2016 (and easy to figure out only in retrospect)

Some of these understanding require making things very explicit and simple.

Reminder: Hamiltonian interpretation of monopole-instanton in zero 't Hooft flux background

Consider compactifying $\mathbb{R}^3 \times S^1_L$ to $T^2 \times \mathbb{R} \times S^1_L$

First, let me provide a Hamiltonian interpretation of 't Hooft-Polyakov monopole instanton in the absence of 't Hooft flux background. (e.g. Bank's book, page 226). The story I will tell you later will be crucially different from this standard (but not sufficiently well-know) discussion.

A monopole-instanton in the case of Polyakov model **always** changes the energy of vacuum state at finite Area(T^2). If $\boldsymbol{\Phi} = \int_{T^2} \boldsymbol{B} = \frac{2\pi}{g} \boldsymbol{\alpha}_a n_a$ is magnetic flux, then the change in energy between the zero-magnetic flux state and $\boldsymbol{\Phi}$ flux state is:

$$\Delta E = \int_{T^2} \frac{1}{2} \boldsymbol{B}^2 = \frac{1}{2} \frac{\left(\frac{2\pi}{g}\right)^2 n_a^2}{\text{Area}(T^2)} > 0$$

$$\lim_{\text{Area}(T^2) \to \infty} \Delta E = 0$$

These states become degenerate with the zero magnetic flux state.

Turn on 't Hooft flux background in 3-direction

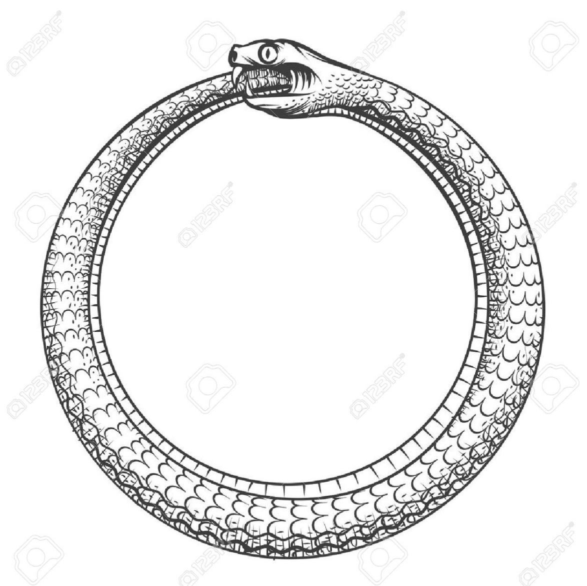
Multiple ways to think about it:

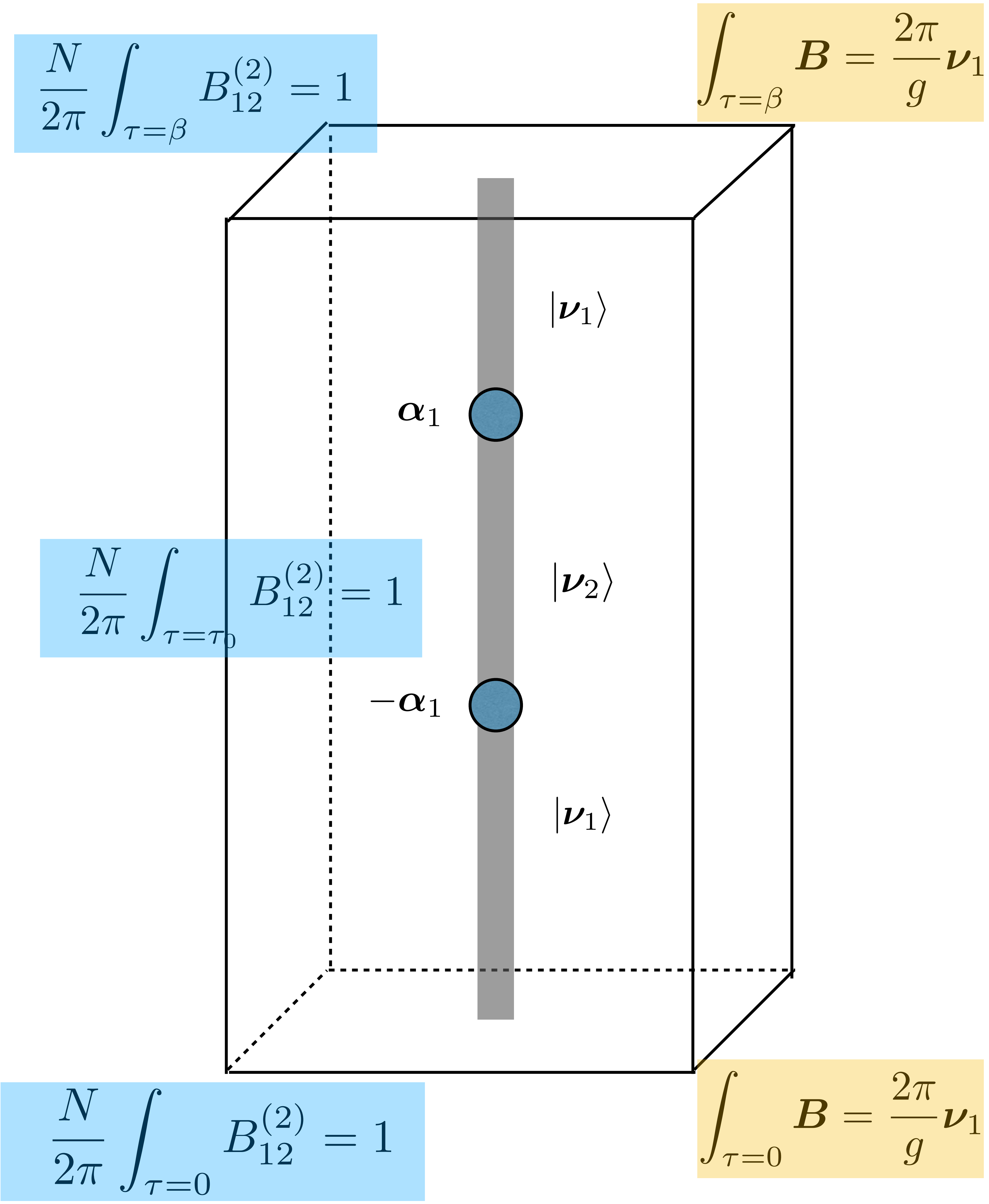
1) Z_N TQFT background.

2) Non-dynamical center-vortex.

3) There can be decorations of center-vortex by dynamical monopoles associated with root lattice which do not change 't Hooft flux, but change magnetic flux through T2. (See Greensite's review, but there vortex is dynamical.)

4) One can think of 1-unit of 't Hooft flux as if it is sourced by fundamental monopole who's charge is in weight lattice. But center-vortex can exist on its own right without any source, and fundamental monopole does not exist in SU(N) theory. If you wish to think in this dangerous description, the center-vortex can be viewed as if a snake eating its own tail. (Ouroboros, from ancient Egypt)





Classification of tunnelings in 't Hooft flux background

But now, there is something more interesting. Consider the following magnetic flux configurations (all of which have the same 't Hooft -flux), which can be connected by monopoles in root lattice.

$$\boldsymbol{\Phi} = \int_{T^2} \boldsymbol{B} = \frac{2\pi}{g}\boldsymbol{\nu}_a, \qquad a = 1, \ldots, N.$$

which are exactly degenerate.

$$E_a = \tfrac{1}{2}\int_{T^2} {\boldsymbol{B}_a}^2 = \frac{1}{2A}\left(\frac{2\pi}{g}\right)^2 \boldsymbol{\nu}_a^2 = \frac{1}{2A}\left(\frac{2\pi}{g}\right)^2 \left(1 - \frac{1}{N}\right), \qquad a = 1, \ldots, N. \qquad E_a - E_b = 0$$

But the rest of all other magnetic flux configurations have higher energy at finite Area(T2) and become only degenerate in the infinite Area(T2) limit.

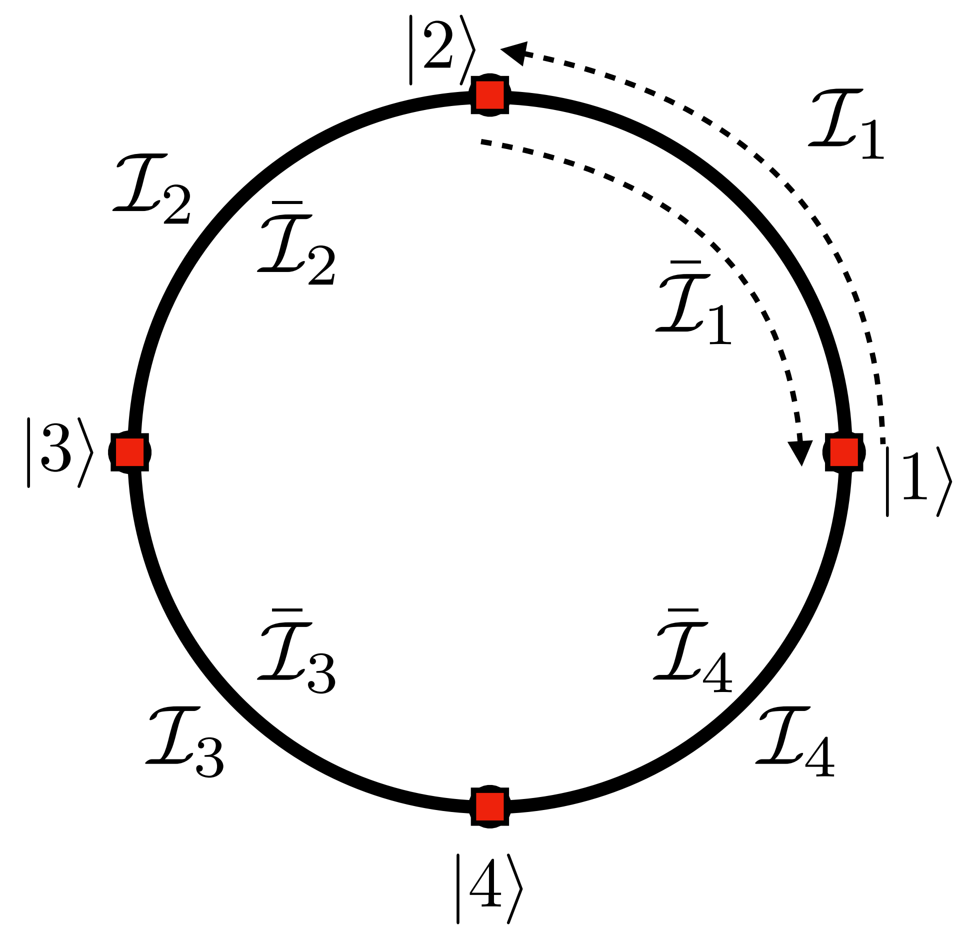
On finite T2 x R x S_L, there are **two types of tunnelings.**

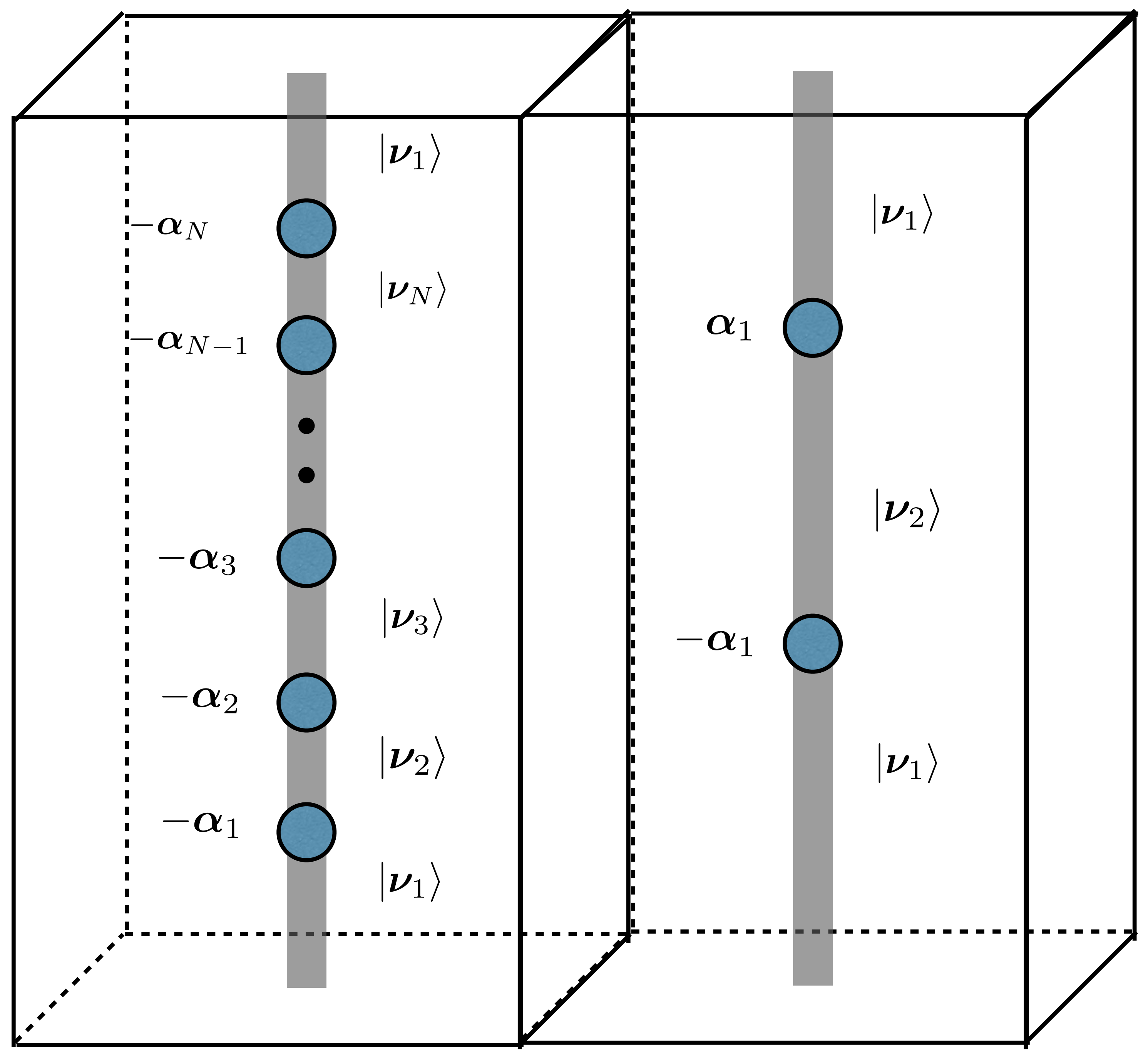
1) Between states that becomes degenerate in Area(T2) tends to infinity limit. Eg. Polyakov, dYM both without TQFT background.
2) Between states that are already degenerate at finite Area(T2). This one is new, in the presence of TQFT background.

Born-Oppenheimer and T_N model

In the small-T_2 limit, and within Born-Oppenheimer approximation, YM with center-symmetric holonomy along S_L reduces to quantum mechanical T_N model!

$$Z_{\ell_{12}} = \mathrm{tr}[\mathrm{e}^{-\beta H_{\ell_{12}}}] = \int_{\mathrm{pbc}} \mathcal{D}a\ \mathrm{e}^{-S(a, B_{12}^{(2)})}$$



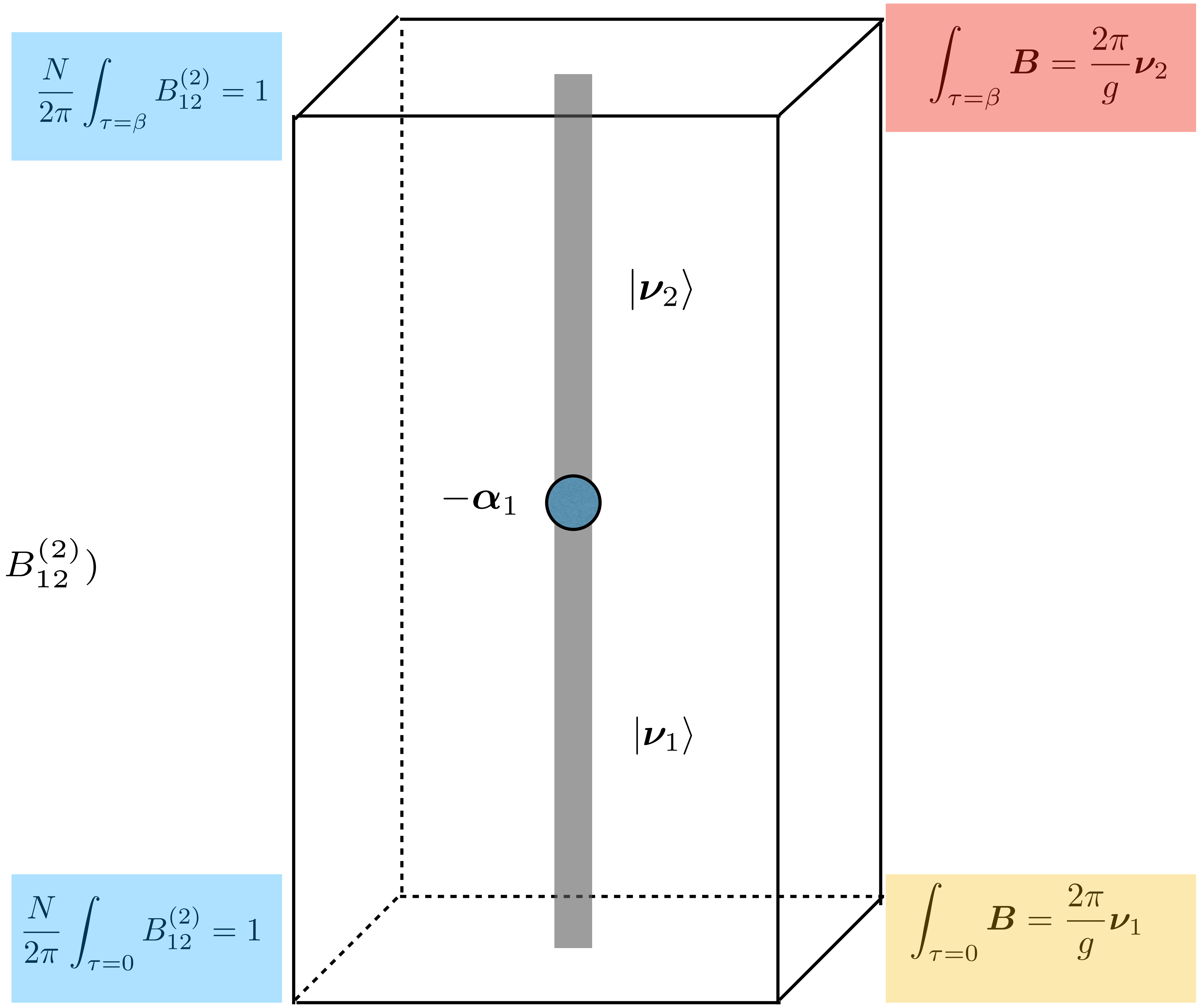


There are N-induced classical minima due to classical Z_N background! In fact, this is one way of phrasing the origin of N-metastable vacua in YM theory!

Hamiltonian description of $\frac{N}{8\pi^2}\int B^{(2)} \wedge B^{(2)} = \frac{\ell_{12}\ell_{34}}{N}$

ℓ_{12} induces a classical potential with N minima.
Sum of transition amplitudes between minima which are ℓ_{34} units apart.

$$
\begin{aligned}
Z_{\ell_{12}\ell_{34}} &= \mathrm{tr}[\mathrm{e}^{-\beta H_{\ell_{12}}}(\mathsf{U}_{\mathrm{c}})^{\ell_{34}}] \\
&= \sum_{a=1}^{N} \langle \boldsymbol{\nu}_a | \mathrm{e}^{-\beta H_{\ell_{12}}}(\mathsf{U}_{\mathrm{c}})^{\ell_{34}} | \boldsymbol{\nu}_a \rangle \\
&= \sum_{a=1}^{N} \langle \boldsymbol{\nu}_a | \mathrm{e}^{-\beta H_{\ell_{12}}} | \boldsymbol{\nu}_{a+\ell_{34}} \rangle \\
&= \int_{\Phi(\beta)=\Phi(0)+\frac{2\pi}{\mathrm{g}}\boldsymbol{\alpha}_{\mathrm{a},\mathrm{a}+\ell_{34}}} \mathcal{D}a\ \mathrm{e}^{-S(a,B^{(2)}_{12})} \\
&= \int_{\mathrm{pbc}} \mathcal{D}a\ \mathrm{e}^{-S(a,B^{(2)}_{12},B^{(2)}_{34})}
\end{aligned}
$$



Here comes the heart of the matter.

Why topological charge and action 1/N? There seems to be 2 unrelated answers!

In center − symmetric background, $\mathrm{Diag}(U_3) = \mathrm{e}^{\mathrm{i}\boldsymbol{\phi}_\star} = (1, \omega, \omega^2, \ldots, \omega^{N-1}),$

$$S_a = \frac{4\pi}{g^2}(\boldsymbol{\alpha}_a.\boldsymbol{\phi}_\star) = \frac{8\pi^2}{g^2 N}, \qquad Q = \frac{1}{2\pi}(\boldsymbol{\alpha}_a.\boldsymbol{\phi}_\star) = \frac{1}{N}$$

In $\mathbb{Z}_N$ TQFT background:

$$S = \frac{8\pi^2}{g^2}\frac{1}{8\pi^2}\int B^{(2)} \wedge B^{(2)} = \frac{8\pi^2}{g^2 N}, \qquad Q = \frac{N}{8\pi^2}\int B^{(2)} \wedge B^{(2)} = \frac{\ell_{12}\ell_{34}}{N} = \frac{1}{N}$$

Are they really unrelated? Not so clear, some of my friends said they are not for good reasons.

TBC vs. monopole-instantons

TBC (conventionally): $U_3(\beta = L_4) = e^{i\frac{2\pi}{N}\ell_{34}} U_3(0).$

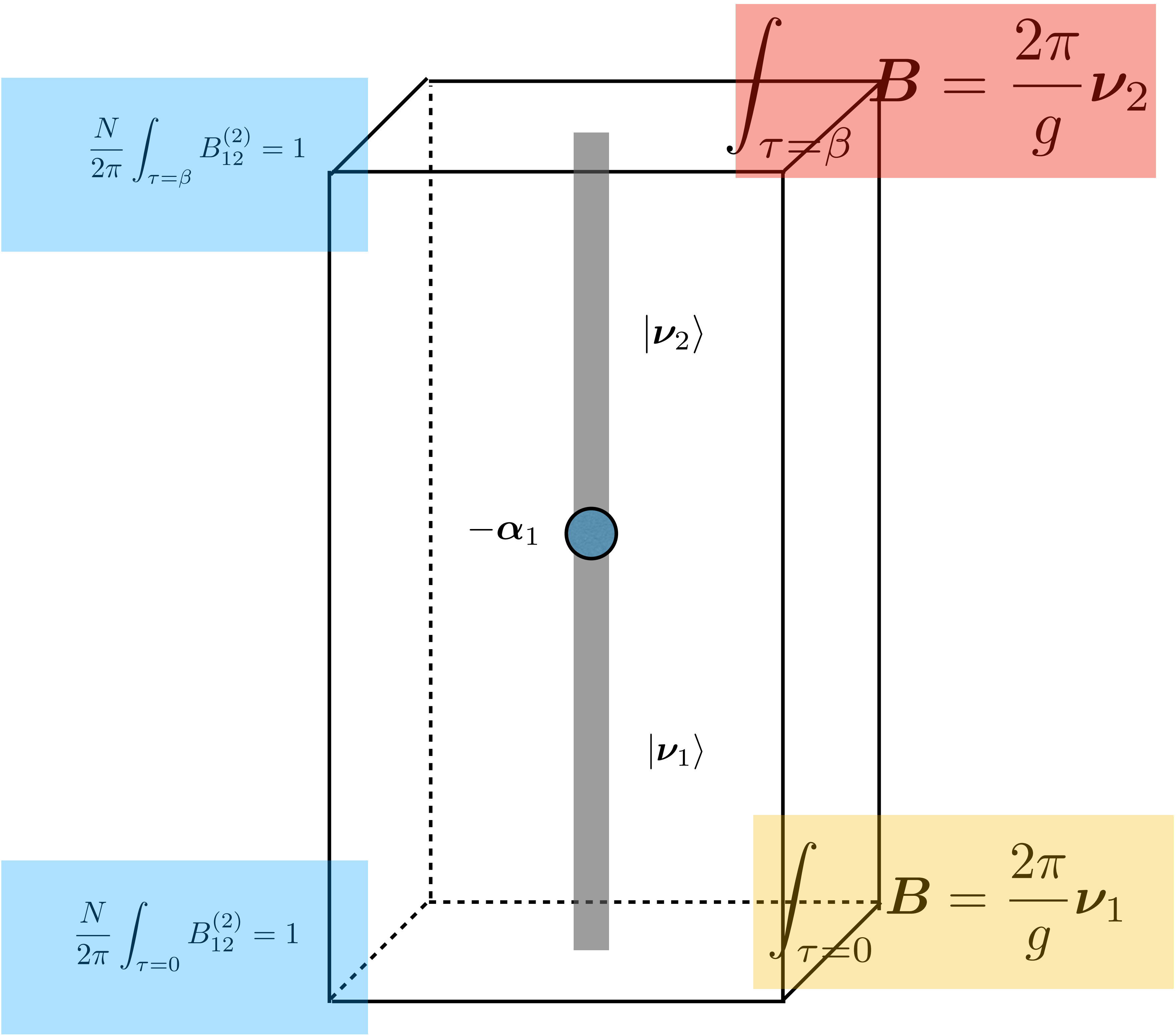
According to Cherman and Poppitz (2016), the gauge invariant rewriting of $U(1)^N$ photon is

$$F_{\mu\nu,k} = \frac{1}{N}\sum_{p=0}^{N-1} e^{-i\frac{2\pi kp}{N}} \mathrm{tr}(U_3^p F_{\mu\nu})$$

and they transform cyclically under zero-form center transformation, and so does dual photons and monopole operators.

Hence, the zero form center-transformation changes the magnetic flux through T2 by a magnetic charge, valued in root lattice. This is our dynamical monopole instanton.

$$\Delta \int_{T^2} \boldsymbol{B} = \frac{2\pi}{g}(\boldsymbol{\nu}_a - \boldsymbol{\nu}_{a+1})$$
$$= -\frac{2\pi}{g}\boldsymbol{\alpha}_a$$



The crucial point here U3 being a center symmetric background! Because of that, center transformation ends up cyclically shifting magnetic flux!

Arbitrary large T3 x large S_L

May be, the results that we obtained in deformed YM on R3 x S1 in 2007 were not some weak coupling, small circle artifacts. May be, they were trying to tell us something deeper about the theory on R4 limit. We thought our construction was not powerful enough, it did not extend to the strong coupling regime and failed us.

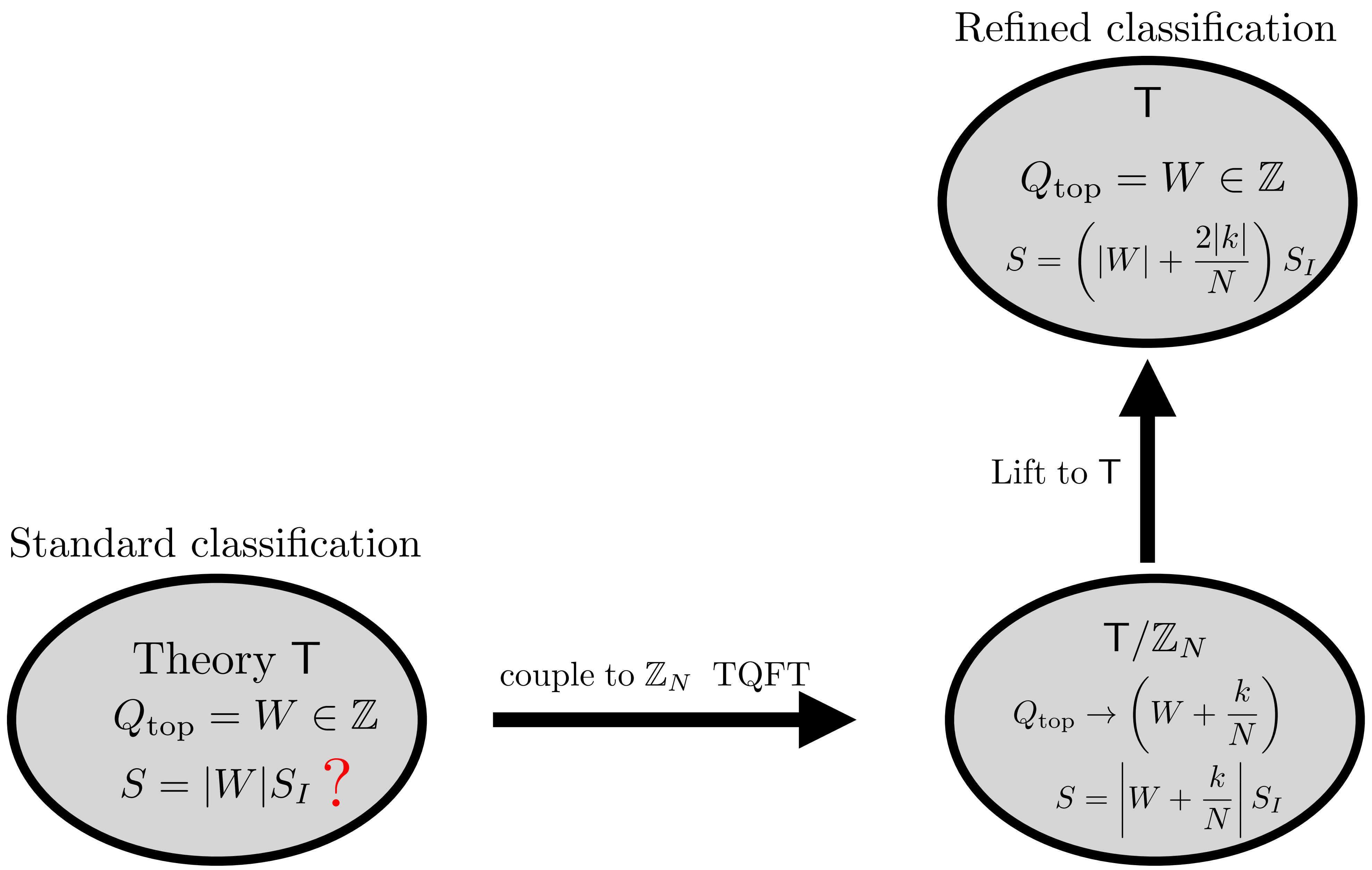
Perhaps, it was other way around. Our theory was much smarter than us, and was trying to guide us towards truth. It was us failing it.

$$S_a = \frac{4\pi}{g^2}(\boldsymbol{\alpha}_a.\boldsymbol{\phi}_\star) = \frac{8\pi^2}{g^2 N}, \qquad Q = \frac{1}{2\pi}(\boldsymbol{\alpha}_a.\boldsymbol{\phi}_\star) = \frac{1}{N}$$

True on arbitrarily large T4 emulating R4.

$$S = \frac{8\pi^2}{g^2}\frac{1}{8\pi^2}\int B^{(2)} \wedge B^{(2)} = \frac{8\pi^2}{g^2 N}, \qquad Q = \frac{N}{8\pi^2}\int B^{(2)} \wedge B^{(2)} = \frac{\ell_{12}\ell_{34}}{N} = \frac{1}{N}$$

We reached to one of our goals. NP expansion on R_4 is controlled by S_I/N, but not S_I. This is not only for YM, but all QCD-like theories and regardless of representations of fermions.



Part 5

Dynamics of QCD(adj) on $\mathbb{R}^3 \times S^1$

Critical points at infinity and bions

QCD(adj) on $\mathbb{R}^3 \times S^1$

$N_f \geq 1$ massless adjoint rep. fermions
periodic boundary conditions ➡ stabilized center symmetry

$$\widetilde{Z}(L) = \mathrm{tr}[e^{-LH}(-1)^F]$$

$$Z = Z_B + Z_F$$
$$\widetilde{Z} = Z_B - Z_F$$

Susy-theory: **Supersymmetric Witten Index, useful.**
Non-susy theory: **Twisted partition function, probably as useful!**

$$V_{1-\mathrm{loop}}[\Omega] = \frac{2}{\pi^2 L^4} \sum_{n=1}^{\infty} \underbrace{\frac{1}{n^4}(-1+N_f)}_{m_n^2} |\mathrm{tr}\,\Omega^n|^2$$

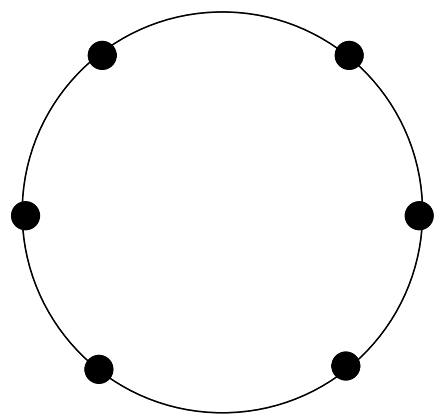
$m_n^2 < 0$ instability, "calculations between 1980-2007"

$m_n^2 = 0$ Supersymmetric case, $N_f = 1$, marginal,

$m_n^2 > 0$ QCD(adj), $N_f > 1$, stability Kovtun, Unsal, Yaffe,07

This sign flip probably gave birth to one of the most promising windows to non-perturbative QCD. Still ongoing work.

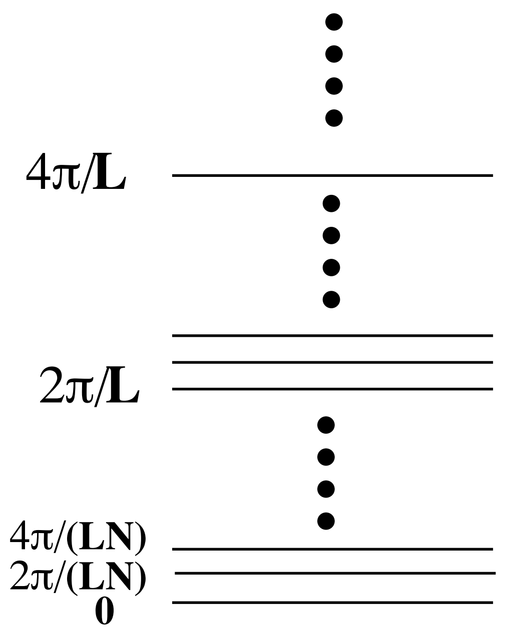
Dynamical abelianization



b)repulsive

unbroken center

$$SU(N) \to U(1)^{N-1}$$



b1)Center−symmetric finite N

Perturbative spectrum, Gapless Cartan subalgebra bosons and fermions.

What happens Non-Pert?

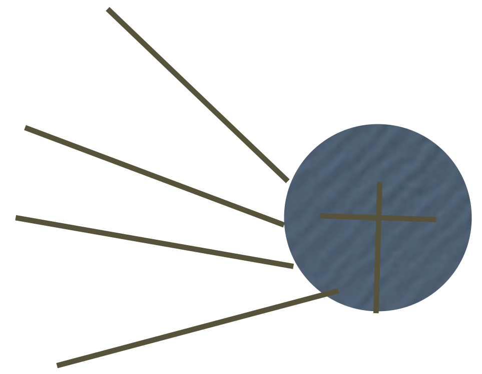
In theories with adjoint fermions?

Theories with massless fermions: take SU(2) QCD(adj)

$$S = \int_{R^3 \times S^1} \frac{1}{g^2} \mathrm{tr} \left[\frac{1}{4} F_{MN}^2 + i \bar{\psi}^I \bar{\sigma}^M D_M \psi_I \right]$$

monopole operators have fermionic zero modes.

$$e^{-S_0} e^{i\sigma} \qquad \underbrace{\psi \ldots \psi}_{\text{fermion zero modes}}$$



Hence, unlike Polyakov mechanism, monopoles can no longer induce mass gap or confinement, instead a photon-fermion interaction Affleck-Harvey-Witten(82).
This is viewed as death of Polyakov mechanism in theories with massless fermions.
AHW proved gaplessness in Polyakov model with Dirac adjoint fermions in 1982 on R3. What happens on R3 x S1?

Is there a gap or not? If so, there must be something new with respect to AHW? How? First, let us count the fermion zero modes.

Index theorems

Journal of Functional Analysis **177**, 203–218 (2000)
doi:10.1006/jfan.2000.3648, available online at http://www.idealibrary.com on IDEAL®

An L^2-Index Theorem for Dirac Operators on $S^1 \times \mathbb{R}^3$

Tom M. W. Nye and Michael A. Singer

Appendix A. Adiabatic limits of η-invariants

$$\begin{aligned} \text{ind}\,(D^+_{\mathbb{A}}) &= \int_X \text{ch}(\mathbb{E}) + \frac{1}{\mu_0}\sum_\mu \epsilon_\mu c_1(E_\mu)[S^2_\infty] \\ &= \int_X \text{ch}(\mathbb{E}) - \frac{1}{2}\overline{\eta}_{\lim} \end{aligned} \tag{22}$$

Very important theorem! Importance of it is not yet sufficiently appreciated in literature.

index theorems

Atiyah-M.I.Singer 1975

Callias 1978 → E. Weinberg 1980

Nye-A.M.Singer, 2000 → Poppitz, MU 2008: The one relevant for us!

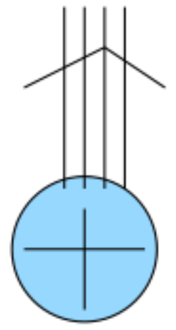
Topological excitations in QCD(adj), SU(2), Nf=2

MÜ 2007

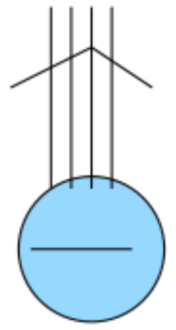
$$\left(\int_{S^2} F,\ \int_{R^3\times S^1} F\tilde{F}\right)$$

index theorems
Callias 1978
E. Weinberg 1980
Nye-A.M.Singer, 2000
Poppitz, MU 2008
Atiyah-M.I.Singer 1975

Monopole-instantons

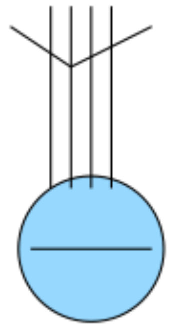

BPS

(1, 1/2)

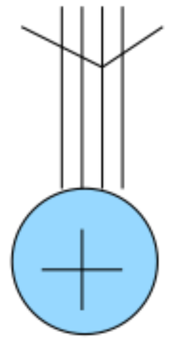

KK

(–1, 1/2)

$$e^{-S_0}e^{i\sigma}\det_{I,J}\psi^I\psi^J,$$


$\overline{\text{BPS}}$

(–1, –1/2)


$\overline{\text{KK}}$

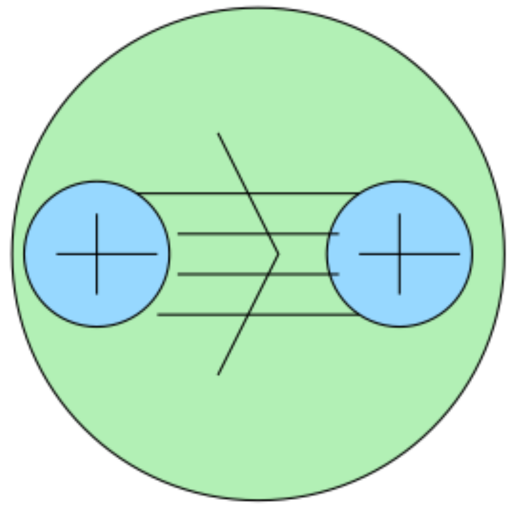
(1, –1/2)

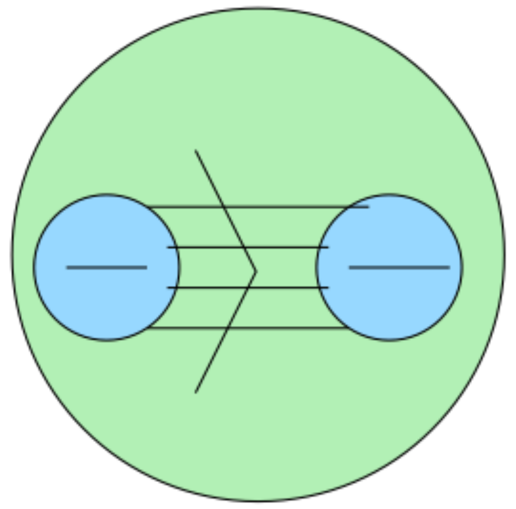
$$e^{-S_0}e^{i\sigma}\det_{I,J}\bar{\psi}^I\bar{\psi}^J$$

Magnetic Bions

Mass gap for gauge fluctuations!

$(\mathbb{Z}_2)_*$


(2,0)


(–2, 0)

No net topological charge.

$$e^{-2S_0}\left(e^{2i\sigma}+e^{-2i\sigma}\right)$$

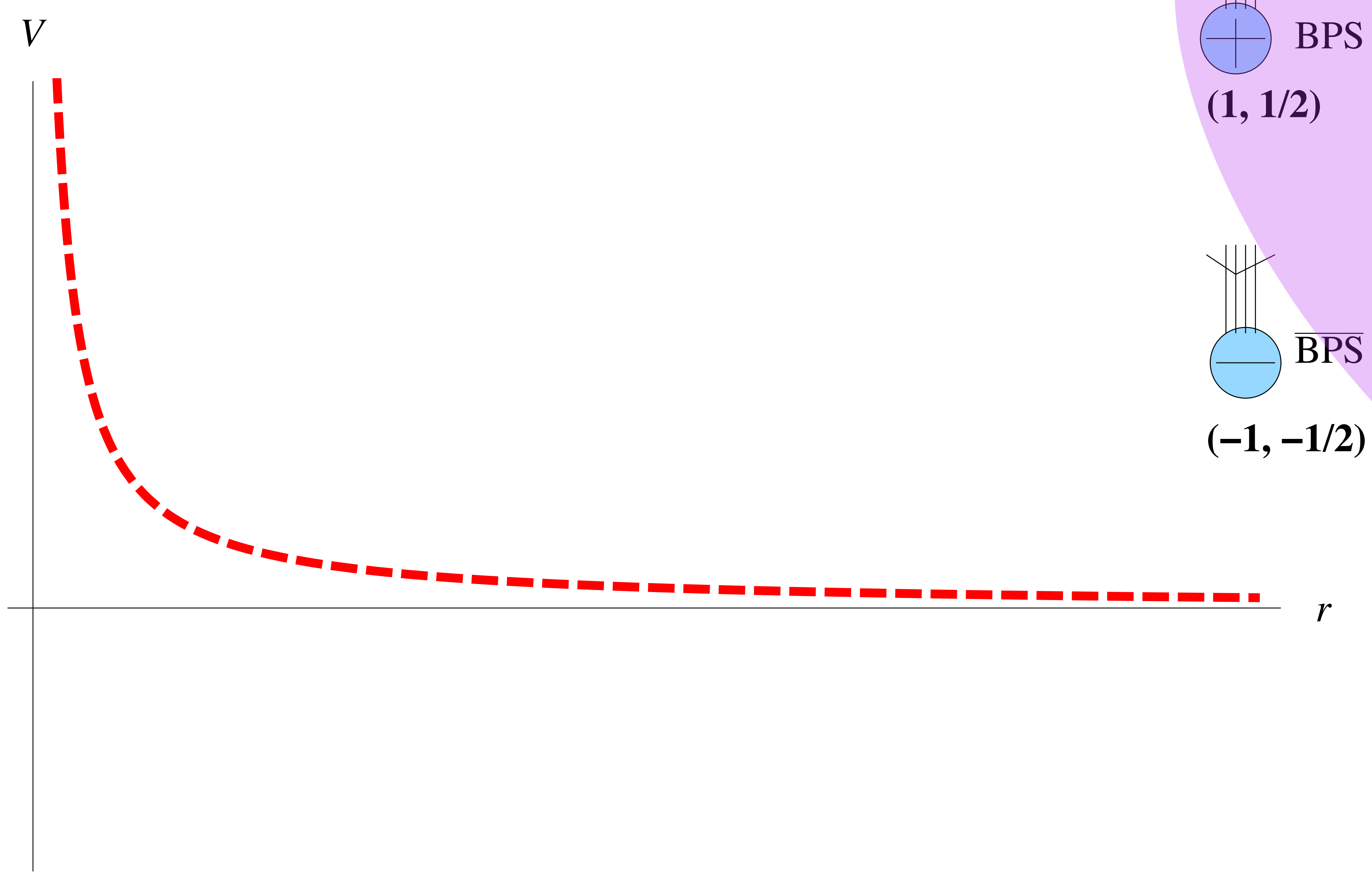
Discrete shift symmetry : $\quad \sigma \longrightarrow \sigma+\pi \qquad \psi^I \longrightarrow e^{i\frac{2\pi}{8}}\psi^I$

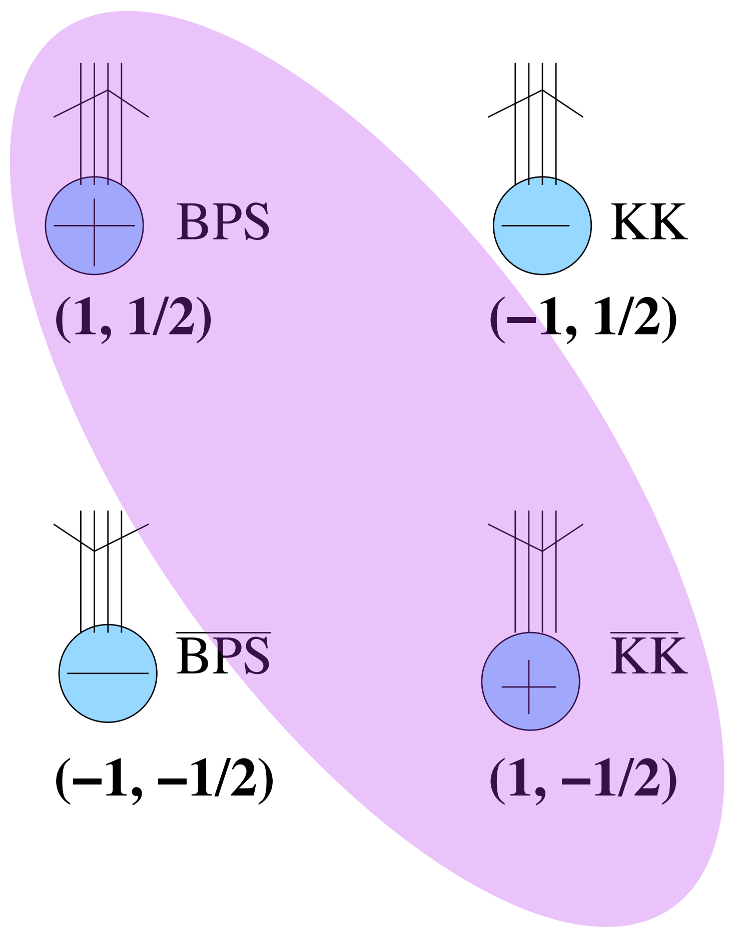
Crucial earlier work: van Baal, Kraan 97/98 and Lee, Lu, Yi, 97/98

Perspective from 2007: Topological molecules

The quantum numbers associated with $e^{-2S_0}(e^{2i\sigma} + e^{-2i\sigma})$ are (2, 0) and (-2,0). Since (2,0) = (1, 1/2) + (1,-1/2), we may think of it as a molecule. We refer to it as magnetic bion.

How is a stable molecule possible? Same sign magnetic charge objects should repel each other due to Coulomb law.





Coulomb law: *1/r* repulsion

Topological molecules

The quantum numbers associated with $e^{-2S_0}(e^{2i\sigma} + e^{-2i\sigma})$ are (2, 0) and (-2,0). Since (2,0) = (1, 1/2) + (1,-1/2), we may think of it as a molecule. We refer to it as magnetic bion.

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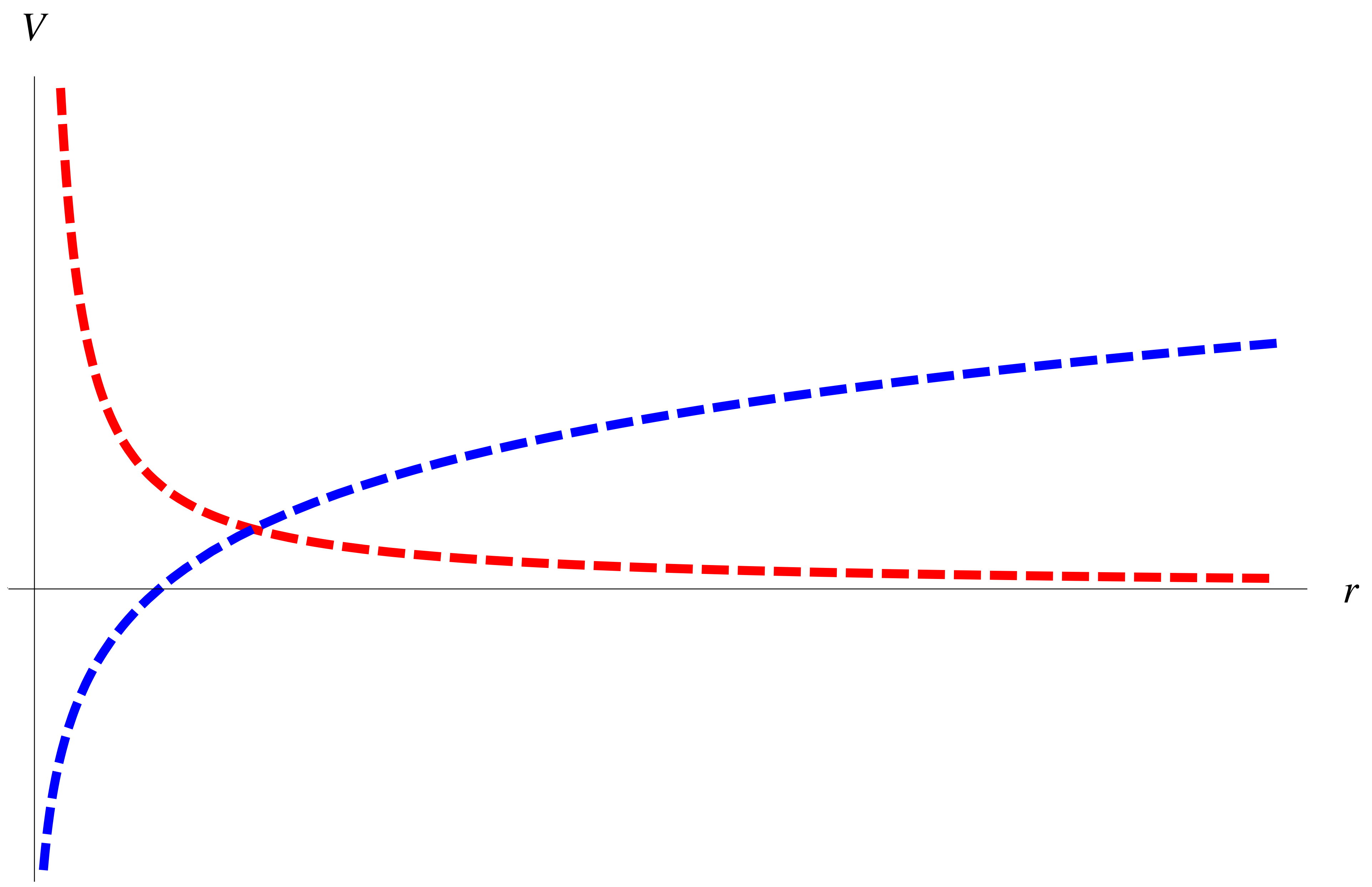

Fermion zero mode exchange:
log(r) attraction.

Coulomb law: *1/r* repulsion

Topological molecules

The quantum numbers associated with $e^{-2S_0}(e^{2i\sigma} + e^{-2i\sigma})$ are (2, 0) and (-2,0). Since (2,0) = (1, 1/2) + (1,-1/2), we may think of it as a molecule. We refer to it as magnetic bion.

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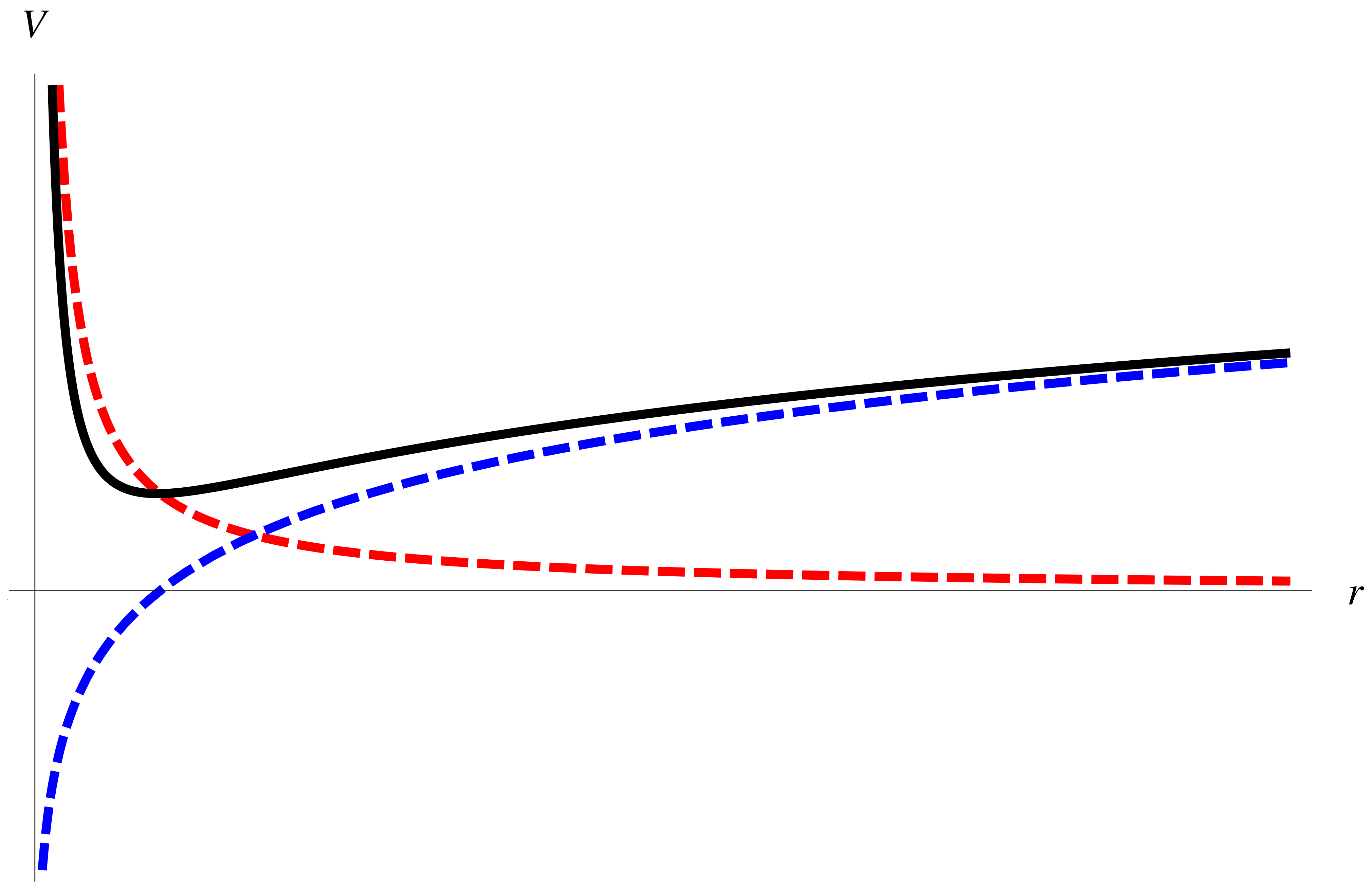


Note: same plot as in QM with Nf fermions

Fermion zero mode exchange: *log(r)* attraction.

Coulomb law: *1/r* repulsion

Sum has a unique minimum.

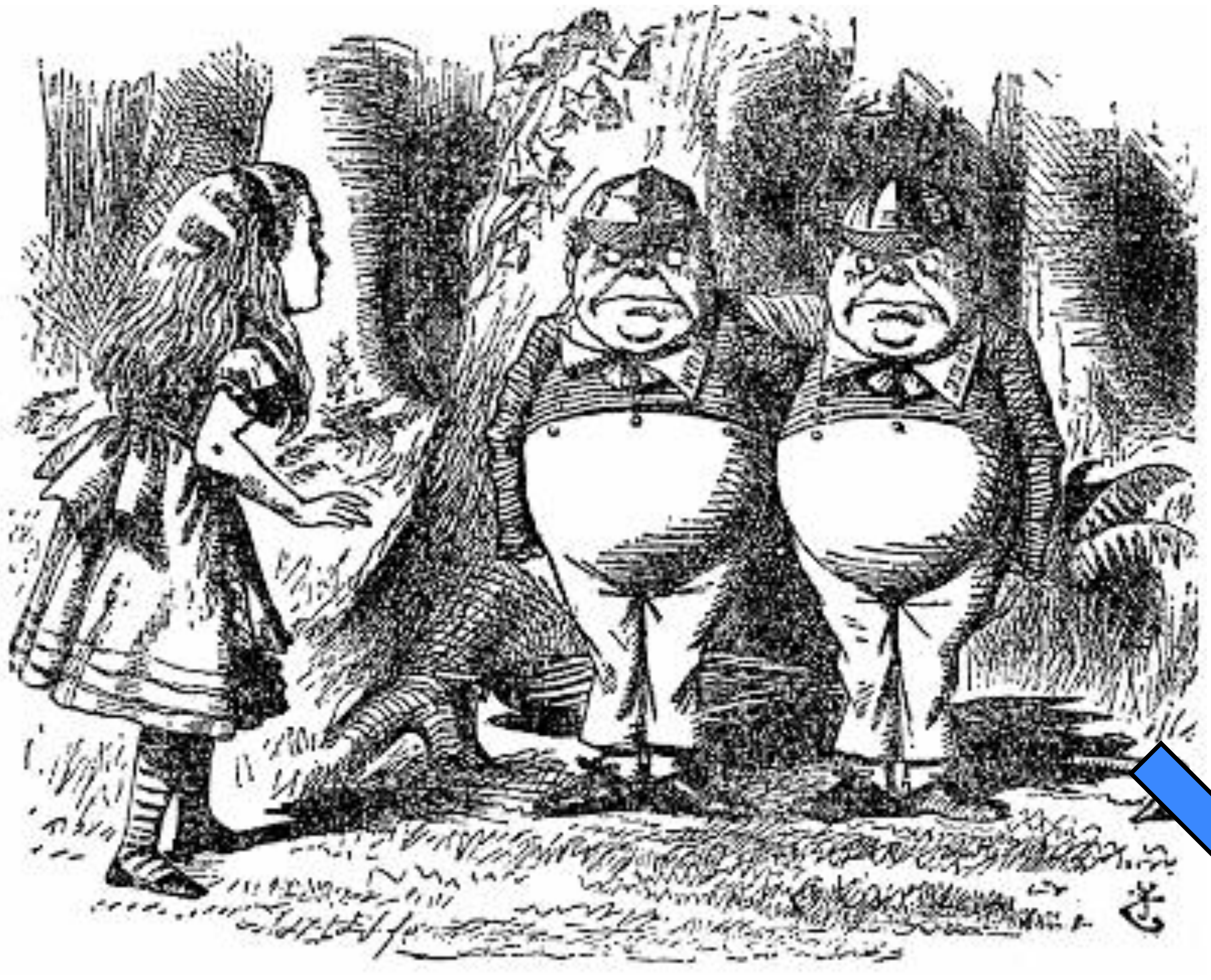
Stable molecules with sizes parametrically larger than monopoles!

QCD(adj) vacuum is a plasma of magnetic bions

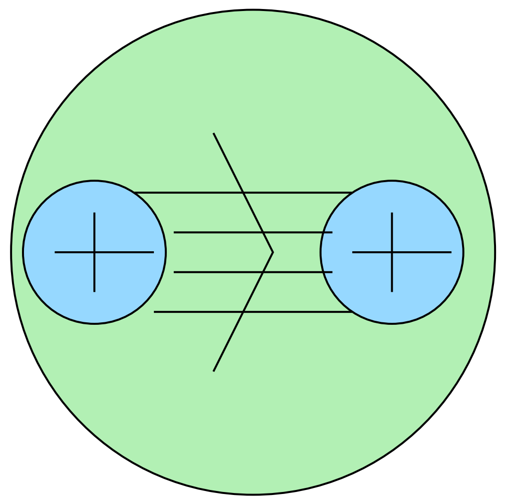
$$L^{\mathrm{dQCD}} = \frac{1}{2}(\partial\sigma)^2 - b\, e^{-2S_0} \cos 2\sigma + i\bar{\psi}^I \gamma_\mu \partial_\mu \psi_I + c\, e^{-S_0} \cos\sigma \big(\det_{I,J} \psi^I \psi^J + \text{c.c.}\big)$$

magnetic bions lead to mass gap!

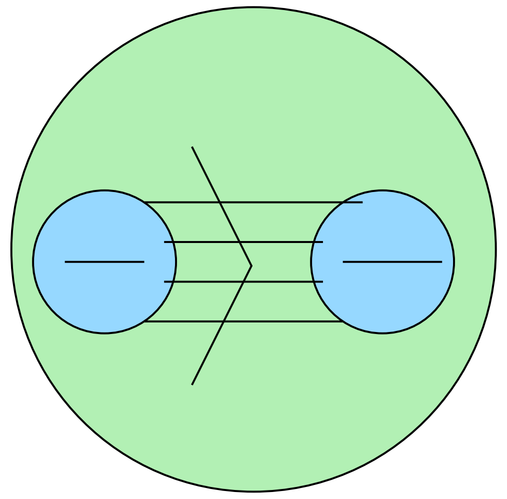
magnetic monopoles



Alice with Tweedledum and Tweedledee, Through the Looking-Glass and what Alice found there (1871).



(2,0)



(−2, 0)

No net topological charge!!

Correlated pairs.

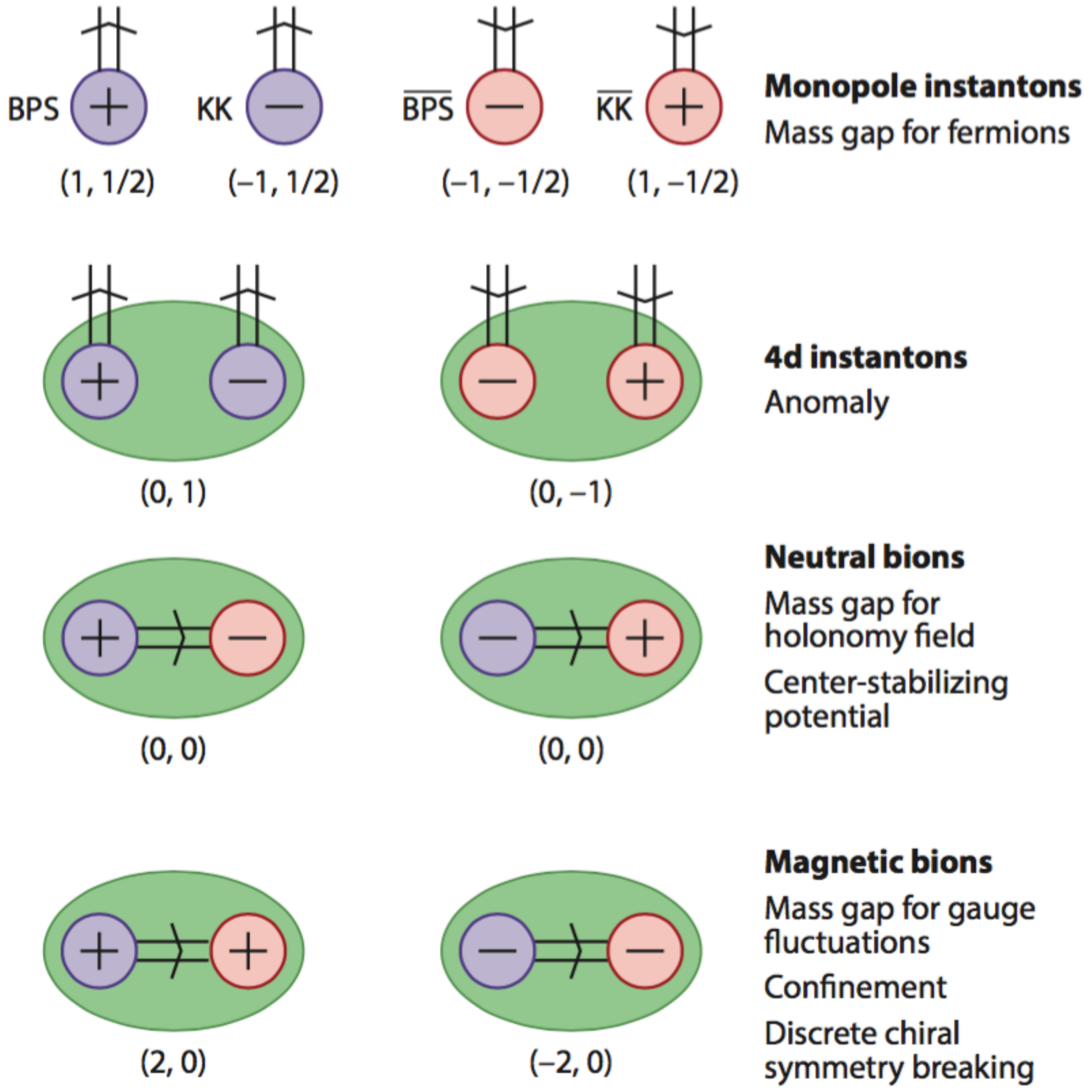
This is the reason why nobody attempted to look for these things.

The first analytic solution for a locally 4d non-susy theory.

Perspective around 2012: Topologically non-trivial and "trivial" saddles

$$(Q_m, Q_{\text{top}}) = \left(\int_{S^2} B \cdot d\Sigma, \int_{\mathbb{R}^3 \times S^1} F\tilde{F} \right)$$

Figure for SU(2)

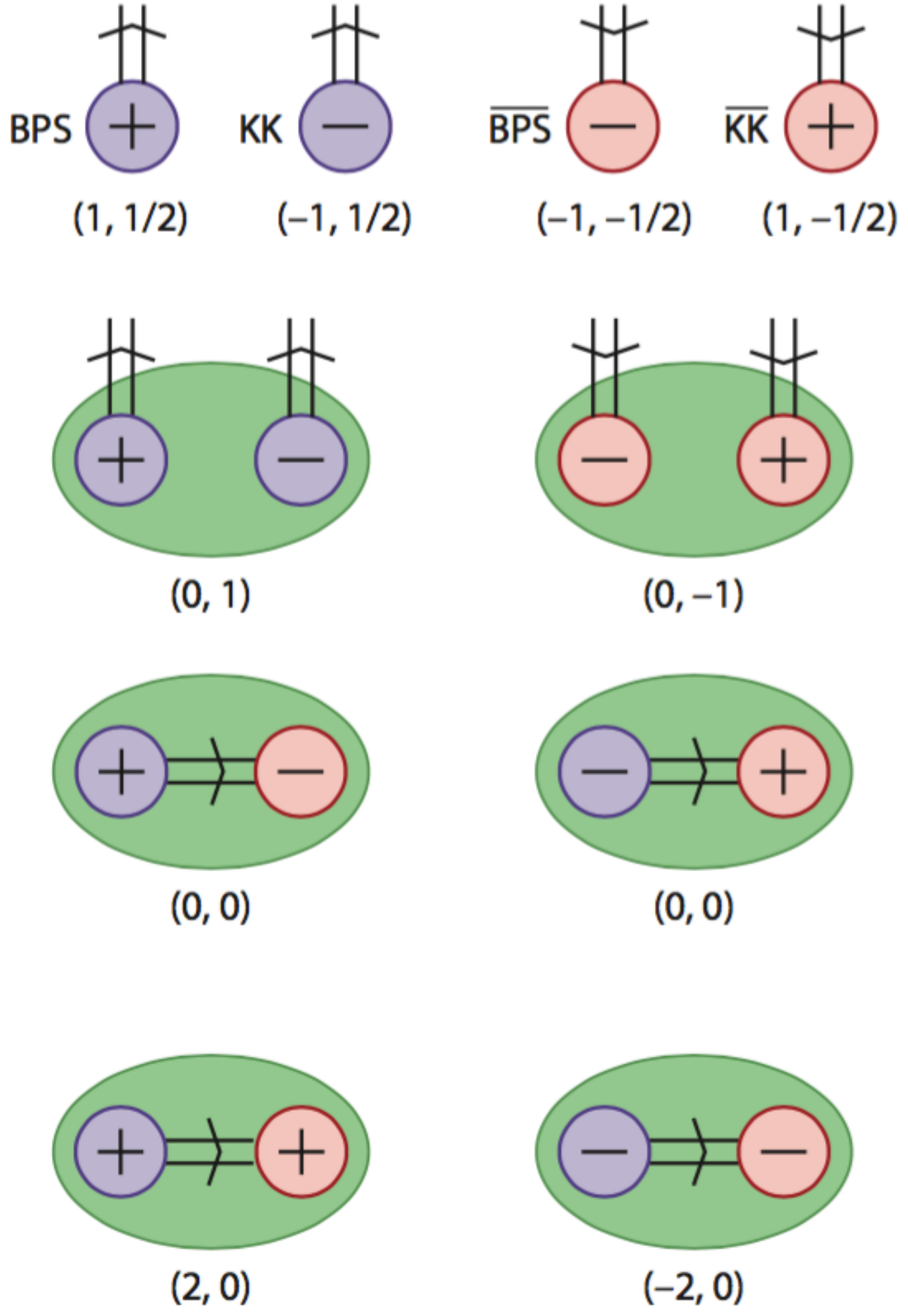


Lesson: Usual topology insufficient to classify saddles in the problem!

Topologically non-trivial and "trivial" saddles

$$(Q_m, Q_{\rm top}) = \left(\int_{S^2} B \cdot d\Sigma, \int_{\mathbb{R}^3 \times S^1} F\tilde{F} \right)$$

Operators for SU(N)



Monopole instantons
Mass gap for fermions

$$e^{-\frac{8\pi^2}{g^2 N} + i\frac{\theta}{N}} e^{-\alpha_i \cdot (b - i\sigma)} (\alpha_i \cdot \lambda)^2$$

4d instantons
Anomaly

$$e^{-\frac{8\pi^2}{g^2} + i\theta} (\lambda\lambda)^N$$

Neutral bions
Mass gap for holonomy field
Center-stabilizing potential

$$e^{-2\frac{8\pi^2}{g^2 N} + i\pi} e^{-2\alpha_i \cdot b}$$

Magnetic bions
Mass gap for gauge fluctuations
Confinement
Discrete chiral symmetry breaking

$$e^{-2\frac{8\pi^2}{g^2 N}} e^{-(\alpha_i + \alpha_{i+1}) \cdot b} e^{i(\alpha_i - \alpha_{i+1}) \cdot \sigma}$$

Lesson: Usual topology insufficient to classify saddles in the problem!
π in neutral bion (Hidden topological angle). It took about 10 years to understand it, requires Lefschetz thimbles.

Topological molecules: 2-defects

2-defects are universal, dictated by Cartan matrix of Lie algebra: Charged and neutral bions

- *Magnetic bions:* For each pair (i, j) such that $(\alpha_i, \alpha_j) < 0$, there exists a magnetic bion $[\mathcal{M}_i \overline{\mathcal{M}}_j]$ with magnetic and topological charges

$$(\mu, \nu) = \left(\alpha_i^\vee - \alpha_j^\vee\,,\, \nu^{(i)} - \nu^{(j)}\right), \tag{5.1}$$

associated with an operator in the effective action proportional to

$$\mathcal{B}_{ij} \sim e^{-S_i(\varphi) - S_j(\varphi)} e^{-2\pi i \sigma (\alpha_i^\vee - \alpha_j^\vee)}, \tag{5.2}$$

- *Neutral bions:* For each i there exists a bion $[\mathcal{M}_i \overline{\mathcal{M}}_i]$ with magnetic and topological charges

$$(\mu, \nu) = (0, 0), \tag{5.3}$$

associated with an operator proportional to

$$\mathcal{B}_{ii} \sim e^{-2S_i(\varphi)}. \tag{5.4}$$

Magnetic bion: mass gap for gauge fluctuations, MÜ 2007
Neutral bion generates a center-stabilizing potential:
Poppitz-MÜ 2011, Poppitz-Schäfer-MÜ, Argyres-MÜ 2012,
Poppitz, Anber, Shifman,

Many other interesting works, especially on sigma models in 2d: (See Misumi's talk and Fujimori's talk)
Kanazawa, Misumi (2014), Gonzalez-Arroyo, Garcia Perez, Sastre 2009,
Bruckmann, Wipf,.... 2007, Nitta, Sakai, ... 2004

Modern perspective: Critical point at infinity and configurations on their thimbles.

- As in the case of QM with Grassmann valued fields, the bion configurations should be viewed as being dominant configurations attached to the thimble of the critical points at infinity.

- Very likely, as in QM, bions are exact solutions for the quantum modified equations of motion.

- It is actually possible to map semi-classical discussion of QCD(adj) to a QM with Grassmann fields with a compactification in TQFT background as we did in YM.

- Recall the amplitudes in QM with fermions from the beginning of the talk. And observe the striking similarities.

Recall: Concept of critical point at infinity and non-Gaussian critical points

$$I_+(\zeta,g) \equiv \int_{\Gamma_{\mathrm{QZM}}^{\theta=0^\pm}} d\tau\, e^{-\frac{A}{g}\left(e^{-\tau}+e^{-(\beta-\tau)}\right)} e^{-\zeta\tau}$$

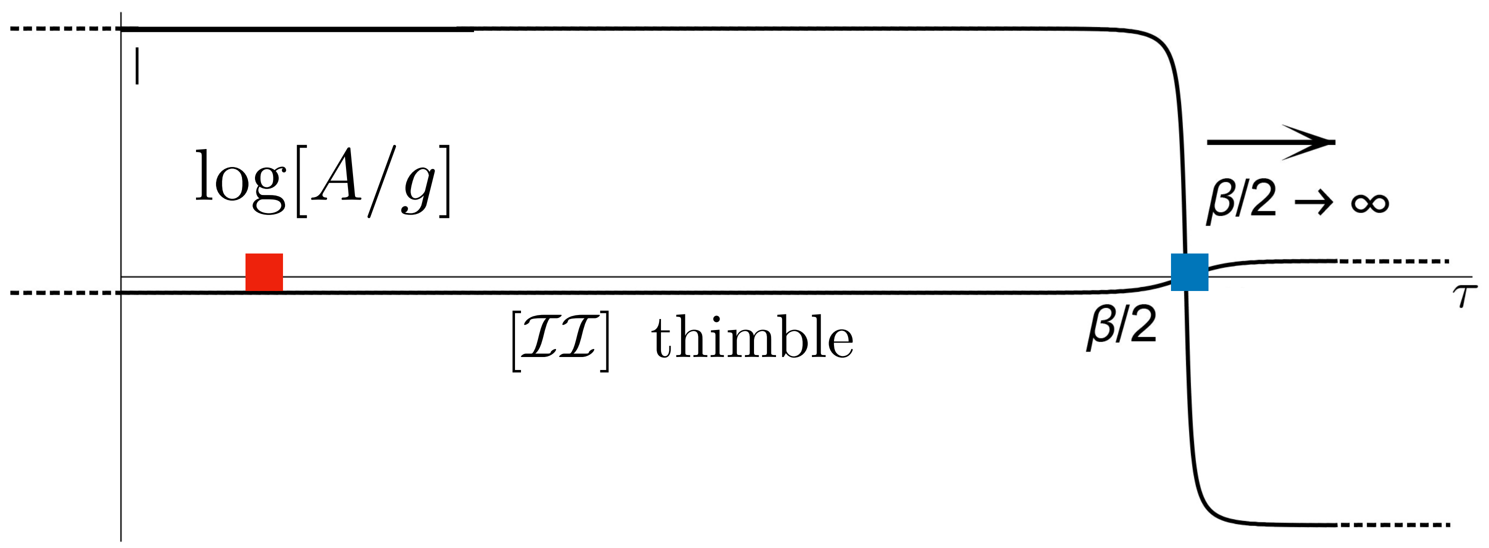
$$I_-(\zeta,g) \equiv \int_{\Gamma_{\mathrm{QZM}}^{\theta=0^\pm}} d\tau\, e^{\frac{A}{g}\left(e^{-\tau}+e^{-(\beta-\tau)}\right)} e^{-\zeta\tau}$$

$$\lim_{\beta\to\infty} e^{-\frac{2A}{g}\left(e^{-\beta/2}\right)} e^{-\zeta\beta/2} = 0\,.$$

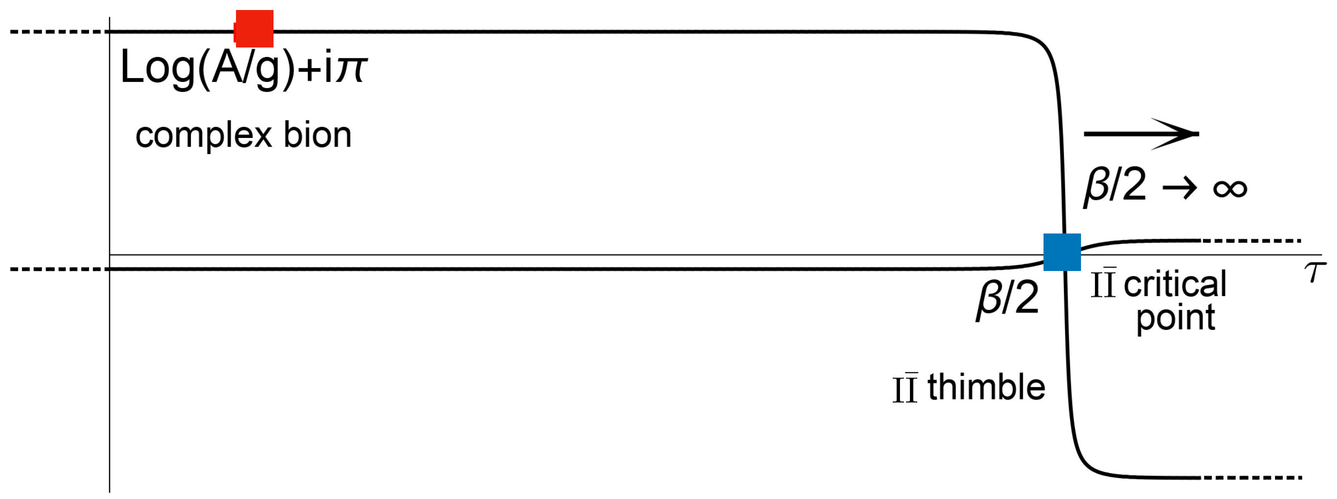
$$\lim_{\beta\to\infty} e^{\frac{2A}{g}\left(e^{-\beta/2}\right)} e^{-\zeta\beta/2} = 0$$

$$\begin{aligned}[\mathcal{I}\mathcal{I}] &= I_+(\zeta,g) \times [\mathcal{I}]^2 \\ &= \left(\frac{g}{A}\right)^{\zeta} \Gamma(\zeta) \times \frac{S_I}{2\pi}\left[\frac{\widehat{\det}\,\mathcal{M}_I}{\det \mathcal{M}_0}\right]^{-1} e^{-2S_I} \\ &= \frac{1}{2\pi}\left(\frac{g}{32}\right)^{\zeta-1} \Gamma(\zeta) e^{-2S_I}\,.\end{aligned}$$

$$\begin{aligned}[\mathcal{I}\bar{\mathcal{I}}]_\pm &= I_-(\zeta,g) \times [\mathcal{I}]^2 \\ &= e^{\pm i\pi\zeta}\left(\frac{g}{A}\right)^{\zeta} \Gamma(\zeta) \times \frac{S_I}{2\pi}\left[\frac{\widehat{\det}\,\mathcal{M}_I}{\det \mathcal{M}_0}\right]^{-1} e^{-2S_I} \\ &= \frac{1}{2\pi}\left(\frac{g}{32}\right)^{\zeta-1} \Gamma(\zeta) e^{-2S_I} e^{\pm i\pi\zeta}\,.\end{aligned}$$



$[\mathcal{I}\bar{\mathcal{I}}]$ thimble



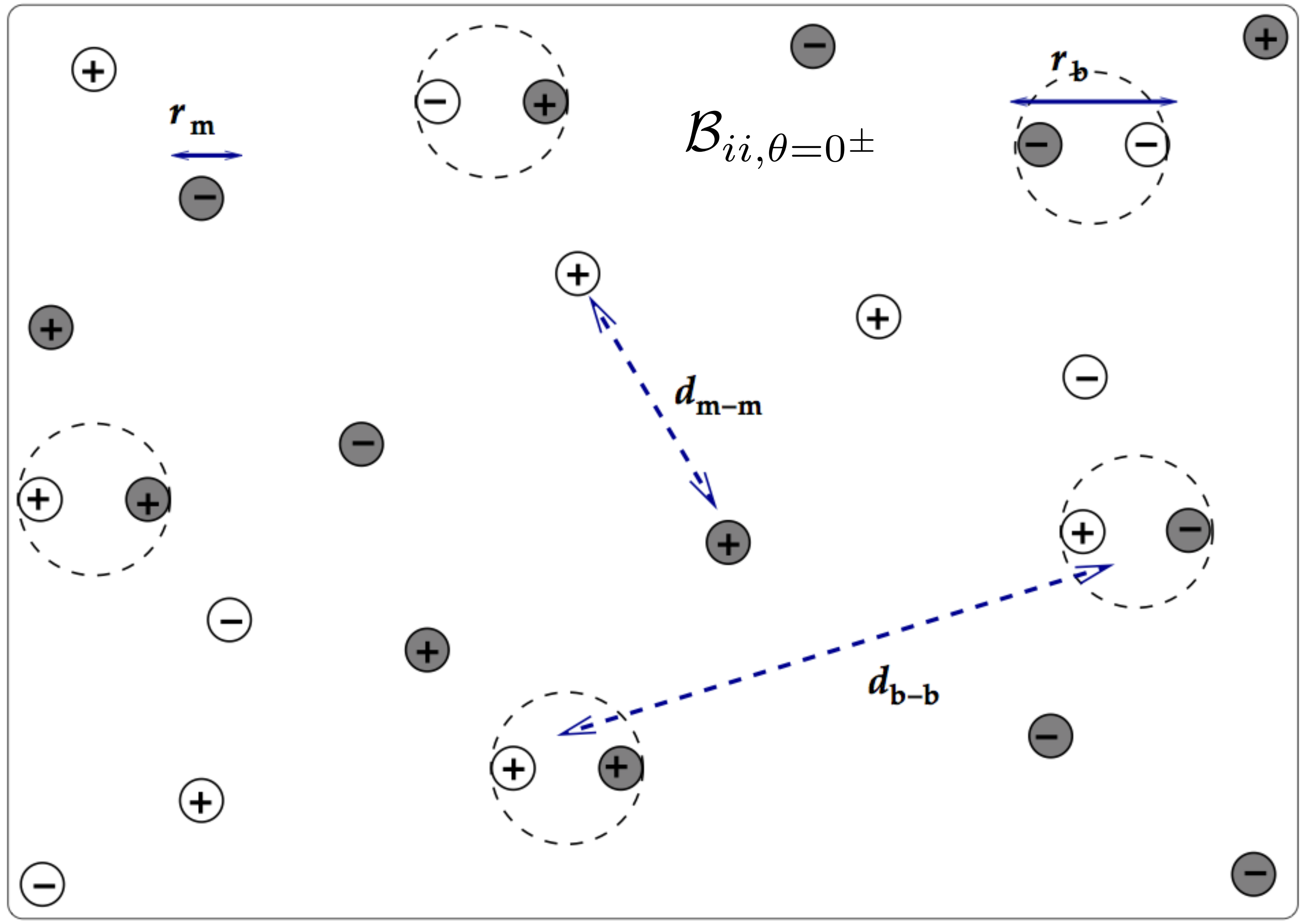
Magnetic/Neutral bion amplitudes and many interesting physical results

$$\mathcal{B}_{12} = [\mathcal{M}_1\overline{\mathcal{M}}_2] = I_{1,\bar{2}} \times \mathcal{M}_1 \times \overline{\mathcal{M}}_2$$
$$= \left(\frac{A}{g^2}\right)^{3-4n_f} \Gamma(4n_f - 3)\mathcal{M}_1 \times \overline{\mathcal{M}}_2$$

$$\mathcal{B}_{11,\pm} = [\mathcal{M}_1\overline{\mathcal{M}}_1]_\pm = I_{1,\bar{1},\pm} \times \mathcal{M}_1 \times \overline{\mathcal{M}}_1$$
$$= \left(\frac{A}{g^2}\right)^{3-4n_f} \Gamma(4n_f - 3)e^{\pm i\pi(3-4n_f)}\, \mathcal{M}_1 \times \overline{\mathcal{M}}_1$$

$$\mathrm{Im}[\mathcal{B}_{11,\pm}] = \pm \left(\frac{A}{g^2}\right)^{3-4n_f} \Gamma(4n_f - 3)\sin(\pi(4n_f - 3))\, \mathcal{M}_1 \times \overline{\mathcal{M}}_1$$
$$= \pm \left(\frac{A}{g^2}\right)^{3-4n_f} \frac{\pi}{\Gamma(4 - 4n_f)} \mathcal{M}_1 \times \overline{\mathcal{M}}_1$$

Deformed YM, Euclidean vacuum

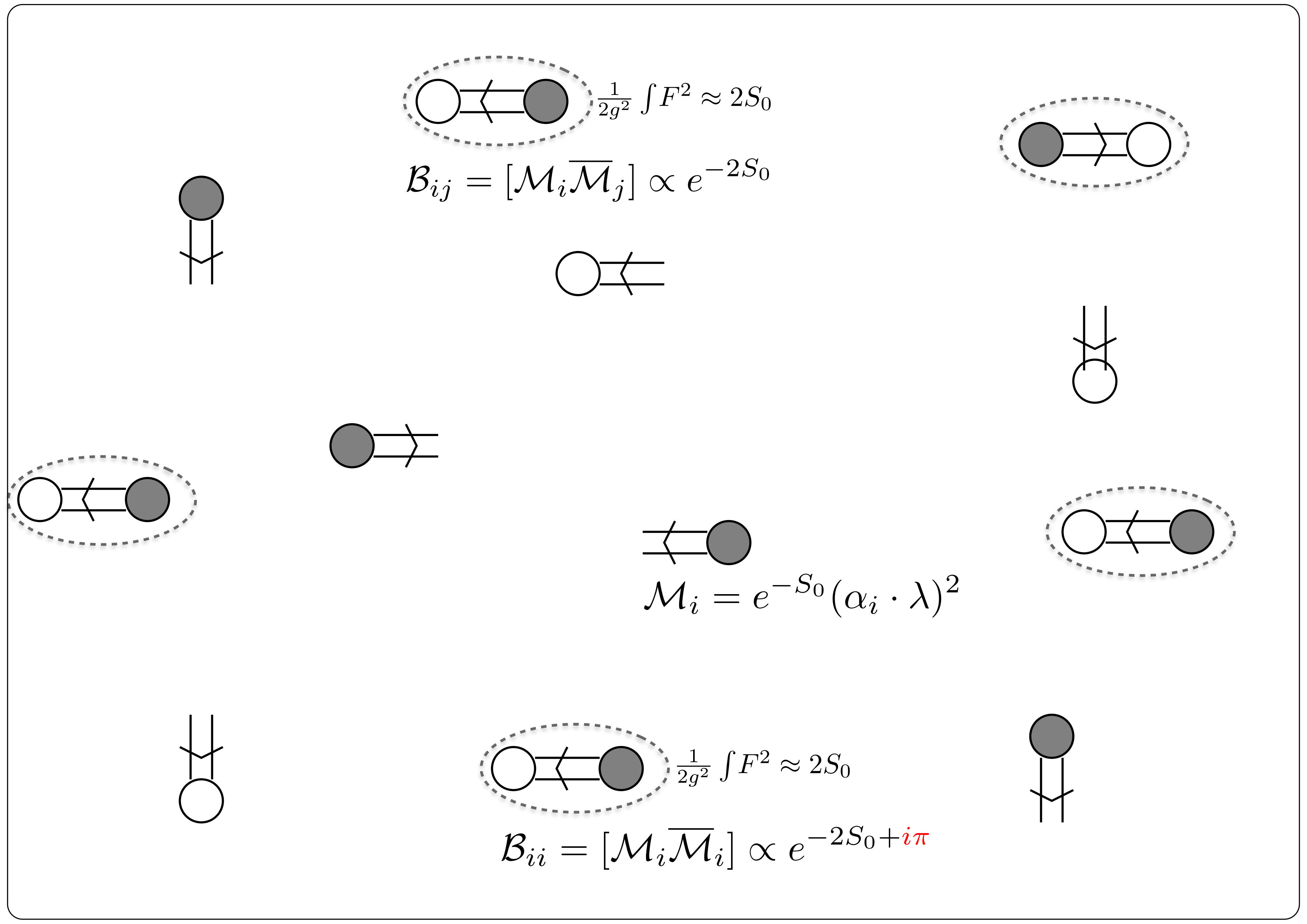


$$\langle F^2 \rangle_{0^\pm} \propto \mathcal{M}_i + [\mathcal{M}_i \bar{\mathcal{M}}_j] + [\mathcal{M}_i \bar{\mathcal{M}}_i]_{0^\pm} + \ldots$$

Ambiguity in gluon condensate sourced by neutral bion.

Relation to R4?

N=1 SYM, Euclidean vacuum



$$\langle F^2 \rangle \propto 0 \times n_{\mathcal{M}_i} + (n_{\mathcal{B}_{ij}} + e^{i\pi} n_{\mathcal{B}_{ii}}) = 0\,.$$

Gluon Condensate vanishes, due to a hidden topological angle. (related to stationary phase associated with thimbles). First micro-realization of a negative contribution to condensate!

IR-Renormalon problem in Yang-Mills theory

't Hooft(79)

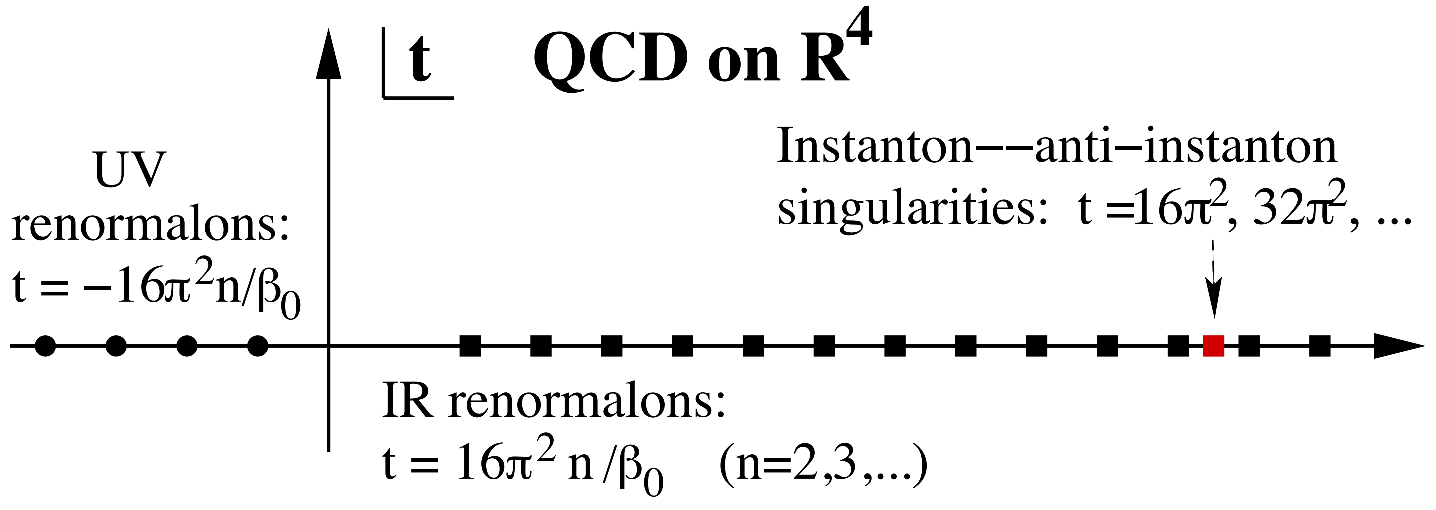
There is a very famous and important problem in Yang-Mills theory, attributed to 't Hooft, which is described in a famous set of lectures "*Can we make sense out of QCD?*"

$[\mathcal{I}\bar{\mathcal{I}}]$ contribution, calculated in some way, gives an ±i exp[-2S_I].
Lipatov(77): Borel-transform BP(t) has singularities at t_n= 2n g^2 S_I.

BUT, BP(t) has other (more important) singularities closer to the origin of the Borel-plane. (not due to factorial growth of number of diagrams, but due to phase space integration.)

't Hooft called these IR-renormalon singularities with the hope that they would be associated with a saddle point like instantons. **No such configuration is known!**

A real problem in QFT, means pert. theory, as is, ill-defined. How to cure starting from microscopic dynamics?



Leading IR singularity $\frac{4S_I}{\beta_0} = \frac{12S_I}{11N}$

Standard view emanating from late 70s

e.g. : from Parisi(78)

If the theory is renormalizable, the Borel transform has new singularities which cannot be controlled by using semi-classical methods [5-8] .

5 G. 't Hooft, Lectures given at Erice (1977)

6 B. Lautrup, Phys. Lett. 69B (1977) 109

7 G. Parisi, Lectures given at the 1977 Cargèse Summer School

8 P. Olesen, Nordita preprint NBI HE 77.48 (1977)

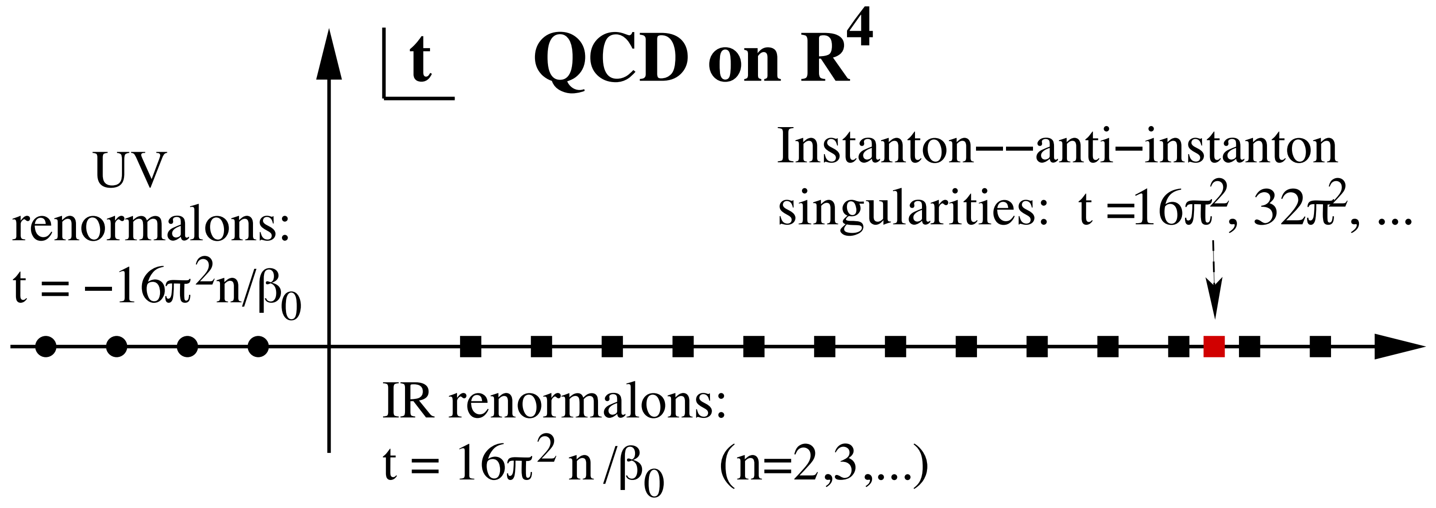
Change the Question: What happens if we can make in deformed Yang-Mills theory in the semi-classically calculable regime?

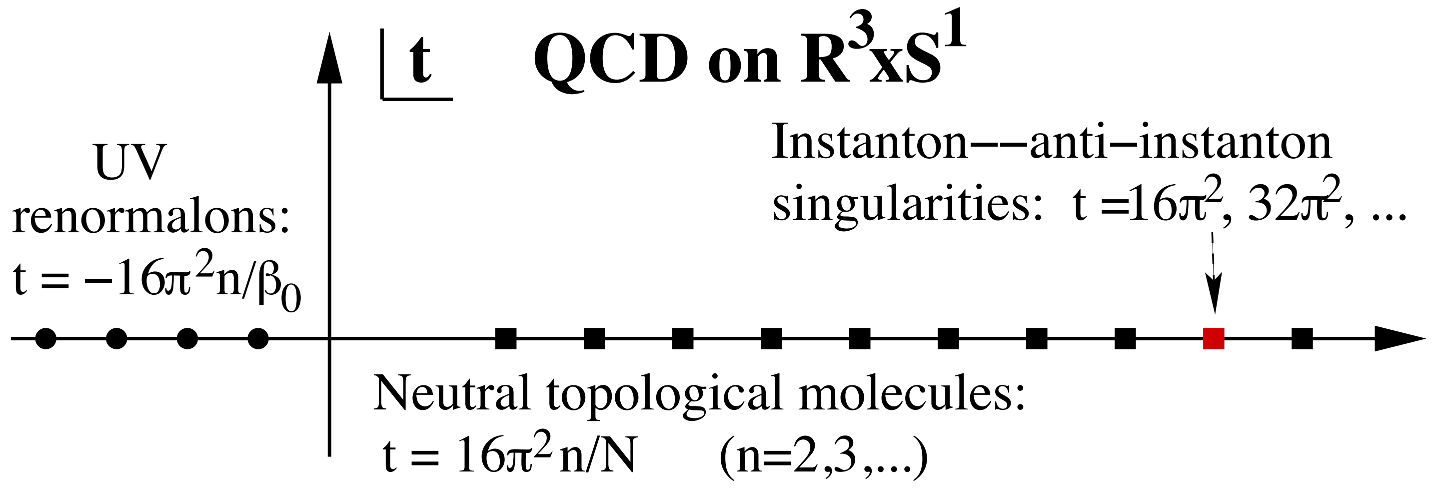
Calculating complex (neutral) bion amplitude similar to QM example:

$$\mathcal{B}_{ii,\theta=0^\pm} = [\mathcal{M}_i \overline{\mathcal{M}}_i]_{\theta=0^\pm}$$

$$\sim e^{-\frac{16\pi^2}{g^2 N}} \pm i e^{-\frac{16\pi^2}{g^2 N}}$$

This corresponds to an IR singularity in the Borel plane at $\frac{2S_I}{N}$





Important thing: 1/N parts match, these singularities in semi-classical domain are avatars of IR-renormalons.
Perhaps, as one moves from weak coupling to strong coupling, 2(S/N) flows to (4S/Beta). Who knows?

Also see Morikawa's talk

Surprises-1

- Many people used to believe that (many still do) confinement, mass gap generation, discrete chiral symmetry breaking, continuous chiral symmetry breaking, topological susceptibilities are **necessarily strong coupling phenomenon** in **4d QCD and QCD-like** theories.

- You can see these statements everywhere both in the old and new literature, and we have seen over the last 13 years that this is complete fallacy.

- We must draw a strict line between **non-perturbative vs. strong coupling** phenomena.

- All of the above are NP phenomena, controlled by $\exp[-c/(Ng^2)]$ effects that can take place both **at weak coupling and strong coupling!**

- Almost all of the known non-trivial strong coupling phenomena can be **continuously connected to weak coupling.**

Results of semi-classical dynamics

- Confinement in QCD(adj) magnetic bion effect, a configuration associated with the thimbles of critical points at infinity.
- Center stability at small-L is a combination of perturbative loop effect and non-perturbative neutral bion effect.
- Discrete chiral symmetry breaking is induced by monopole operators.
- Unique string tension for the quarks in the defining representation (Unlike Polyakov and Seiberg-Witten which admits N-1 types fundamental string tensions).
- Almost all of the known non-trivial strong coupling phenomena can be **continuously connected to weak coupling.**
- **To go to strong coupling, TQFT coupling is very likely useful.**

For a review of some of these ideas, see

New Nonperturbative Methods in Quantum Field Theory: From Large-N Orbifold Equivalence to Bions and Resurgence

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Keywords

large-N orbifold and orientifold equivalence, volume independence, double-trace deformations, adiabatic continuity, semiclassical calculability, magnetic and neutral bions, resurgence, Picard–Lefschetz theory of path integration

Abstract

We present a broad conceptual introduction to some new ideas in nonperturbative quantum field theory (QFT) that have led to progress toward an understanding of quark confinement in gauge theories and, more broadly, toward a nonperturbative continuum definition of QFTs. We first present exact orbifold equivalences of supersymmetric and nonsupersymmetric QFTs in the large-N limit and exact equivalences of large-N theories in infinite volume to large-N theories in finite volume, or even at a single point. We discuss principles by which calculable QFTs are continuously connected to strong-coupling QFTs, allowing understanding of the physics of confinement or the absence thereof. We discuss the role of particular saddle solutions, termed bions, in weak-coupling calculable regimes. The properties of bions motivate an extension of semiclassical methods used to evaluate functional integrals to include families of complex saddles (Picard–Lefschetz theory). This analysis leads us to the resurgence program, which may provide a framework for combining divergent perturbation series with semiclassical instanton and bion/renormalon contributions. This program could provide a nonperturbative definition of the path integral.

Contents

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