Large N and Small N in Yang-Mills Theory

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Based on collaboration with

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To Appear

Motivations

Today:

Pure Yang-Mills Theory

w/ θ-angle w/ $G = SU(N)$

Today :

Pure Yang-Mills Theory

w/ θ-angle w/ $G = SU(N)$

vacuum energy $E(\theta, N) = ?$

expansion around $\theta = 0$

$$E(\theta) - E(0) = \frac{1}{2}\chi\theta^2\left(1 + b_2\theta^2 + b_4\theta^6 + \cdots\right)$$

coefficients

(also motivation from axionic inflation

[Nomura-Watari-Y, Nomura-Y ('17)])

Instanton (DIGA) ['t Hooft]

$$E(\theta) \sim 1 - \cos\theta \rightsquigarrow b_2 = -\frac{1}{12}, \quad b_4 = \frac{1}{360}, \quad \cdots$$

Instanton (DIGA) [t'Hooft]

$$E(\theta) \sim 1-\cos\theta \rightsquigarrow b_2 = -\frac{1}{12}, \quad b_4 = \frac{1}{360}, \quad \cdots$$

Large N [t'Hooft, Witten, ...]

$$\mathcal{L} \sim \frac{1}{N^{-1}}\left(\frac{1}{g^2 N}\,\mathrm{Tr}\,F\wedge *F + \frac{\theta}{N}\,\mathrm{Tr}\,F\wedge F\right)$$

$$\downarrow$$

$$E(\theta) = N^2 f\left(\frac{\theta}{N}\right) = \frac{1}{2}\chi\,\theta^2\left(1 + b_2\theta^2 + \cdots\right)$$

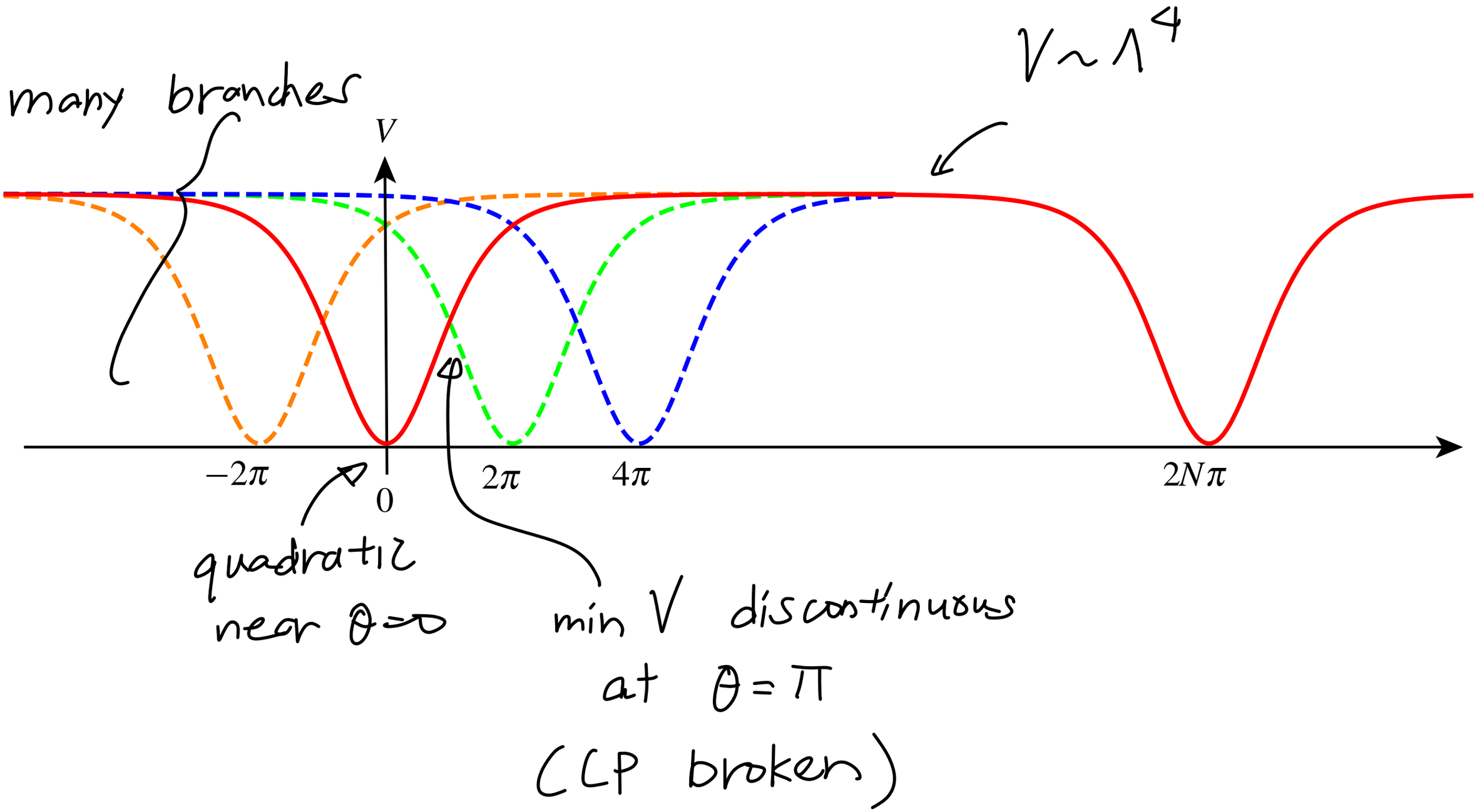
$$\downarrow$$

$$\chi = \chi^{(0)} + O\left(\frac{1}{N^2}\right)$$

$$b_{2n} = \frac{b_{2n}^{(0)}}{N^{2n}} + O\left(\frac{1}{N^{2n+2}}\right)$$

NOT 2π-periodic

Expectation for $E(\theta)$



[Figure from Nomura - Y ('17)]

Instanton

$$Z \sim \int_{\rho\sim 0}^{\rho\sim\infty} \frac{d\rho}{\rho} \frac{1}{\rho^4} e^{-\frac{8\pi}{g^2(\mu)}+i\theta} (\mu\rho)^{11N/3}$$

1-loop running

Instanton

$$Z \sim \int_{\rho\sim 0}^{\rho\sim\infty} \frac{d\rho}{\rho} \frac{1}{\rho^4} e^{-\frac{8\pi}{g^2(\mu)}+i\theta} (\mu\rho)^{\underbrace{11N/3}_{\text{1-loop running}}}$$

$$N \gtrsim N_*^{\text{1-loop}} = \frac{12}{11}$$: IR divergence (IR problem)

Instanton

$$Z \sim \int_{\rho\sim 0}^{\rho\sim\infty} \frac{d\rho}{\rho}\frac{1}{\rho^4} e^{-\frac{8\pi}{g^2(\mu)}+i\theta} (\mu\rho)^{11N/3}$$

1-loop running

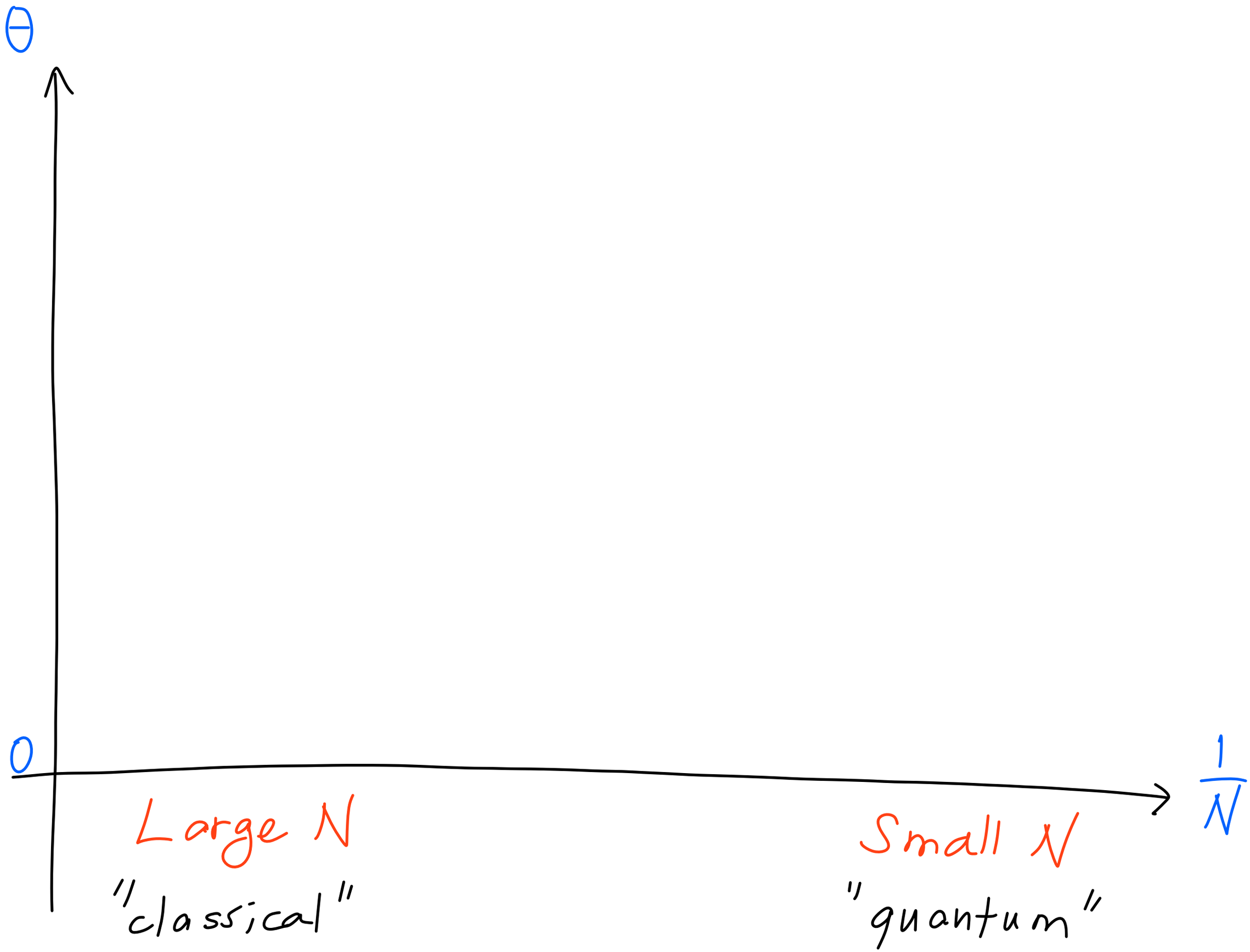
$$\begin{cases} N \gtrsim N_*^{1\text{-loop}} = \frac{12}{11} : \text{IR divergence (IR problem)} \\ N \lesssim N_*^{1\text{-loop}} = \frac{12}{11} : \text{UV divergence } (\rho \gtrsim M^{-1}) \end{cases}$$

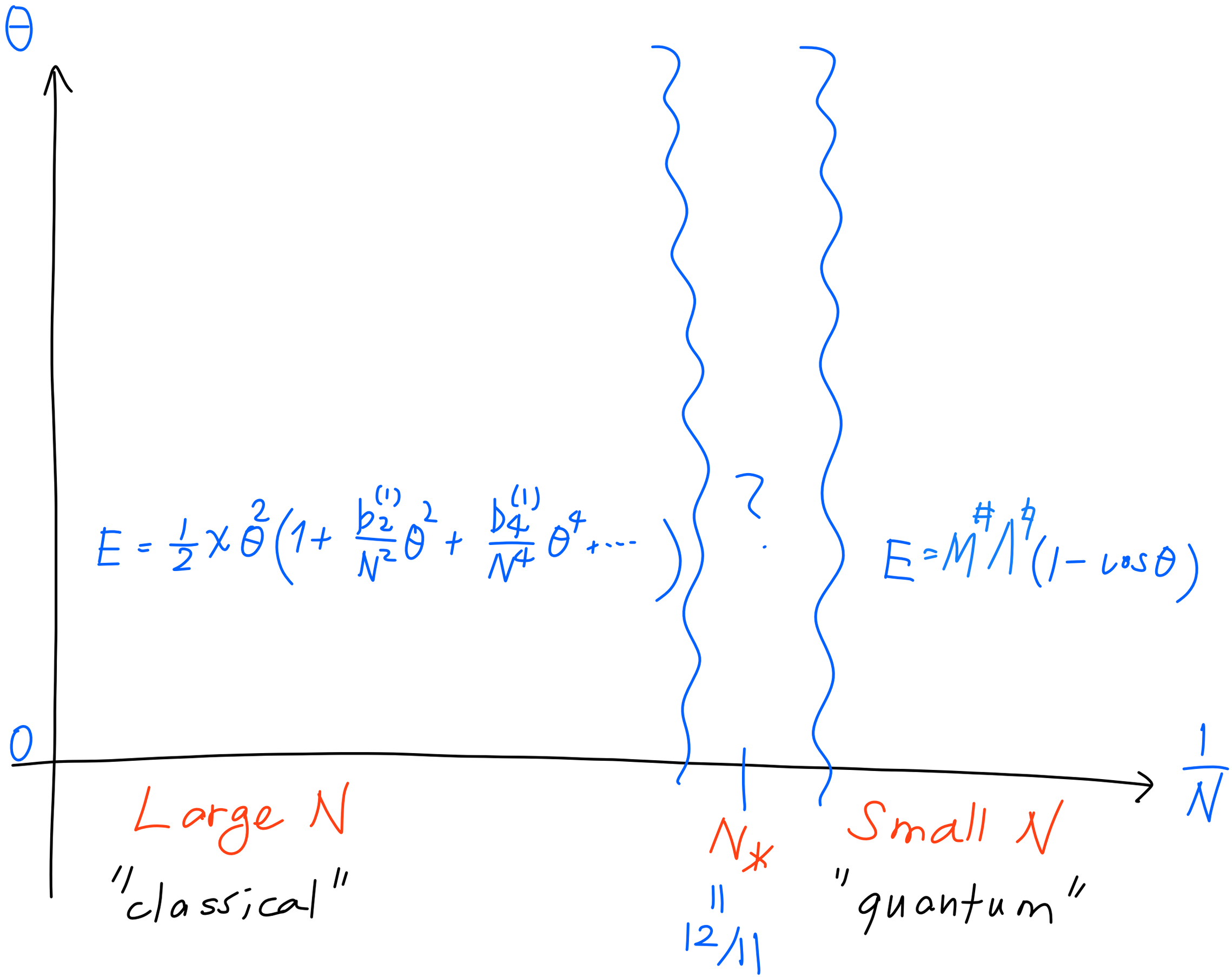
cutoff scale

$$\longrightarrow Z \sim M^{4-\frac{11}{3}N} \Lambda^{\frac{11N}{3}} (1-\cos\theta)$$

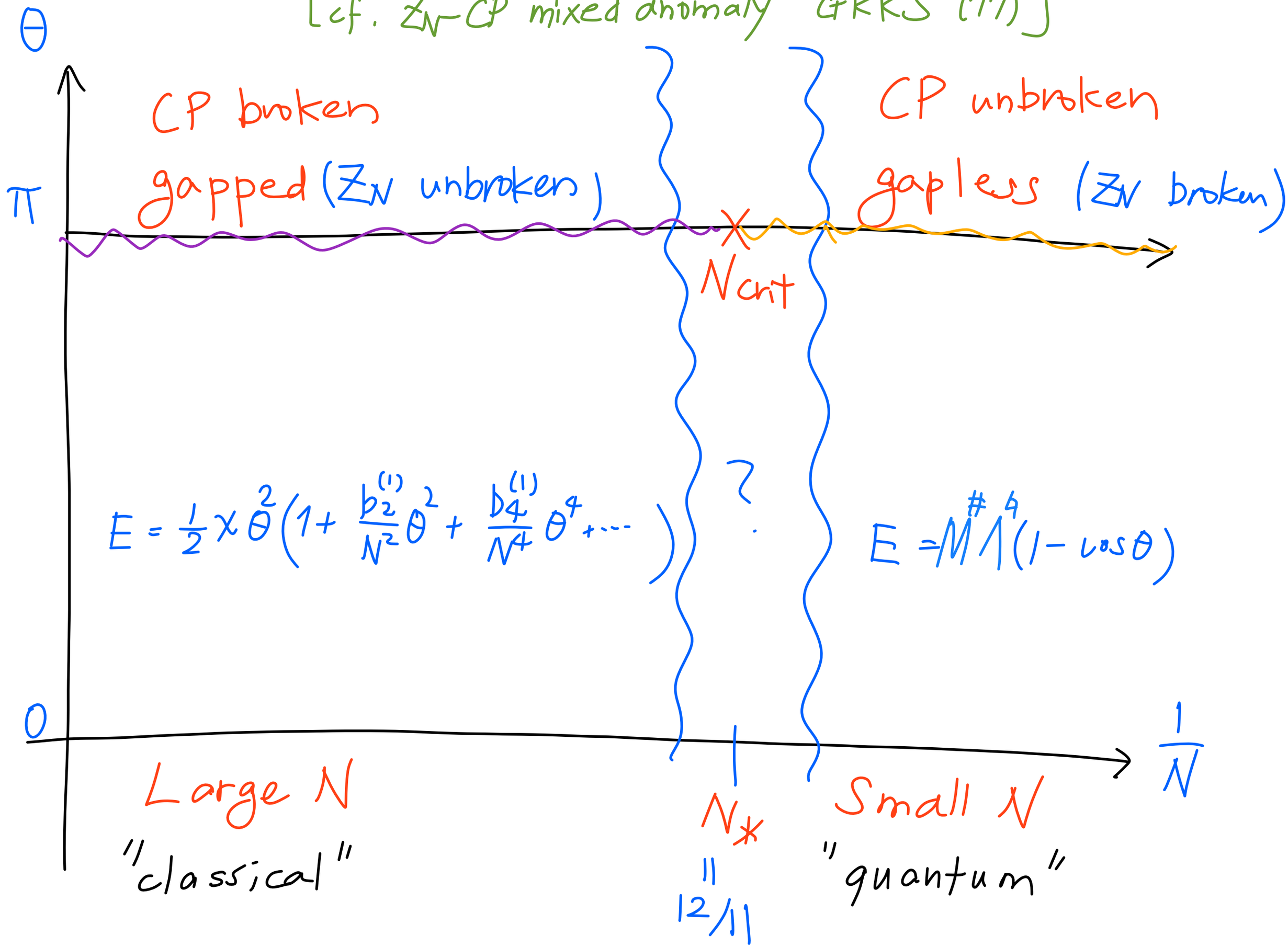
dominates over other contributions

[This happens for $SU(2)_{EW}$ in SM $\rightsquigarrow \Lambda_{c.c}$?
[Nomura-Watari-Yanagida ('00), ⋯, Ibe-Yanagida-Y ('18)]]

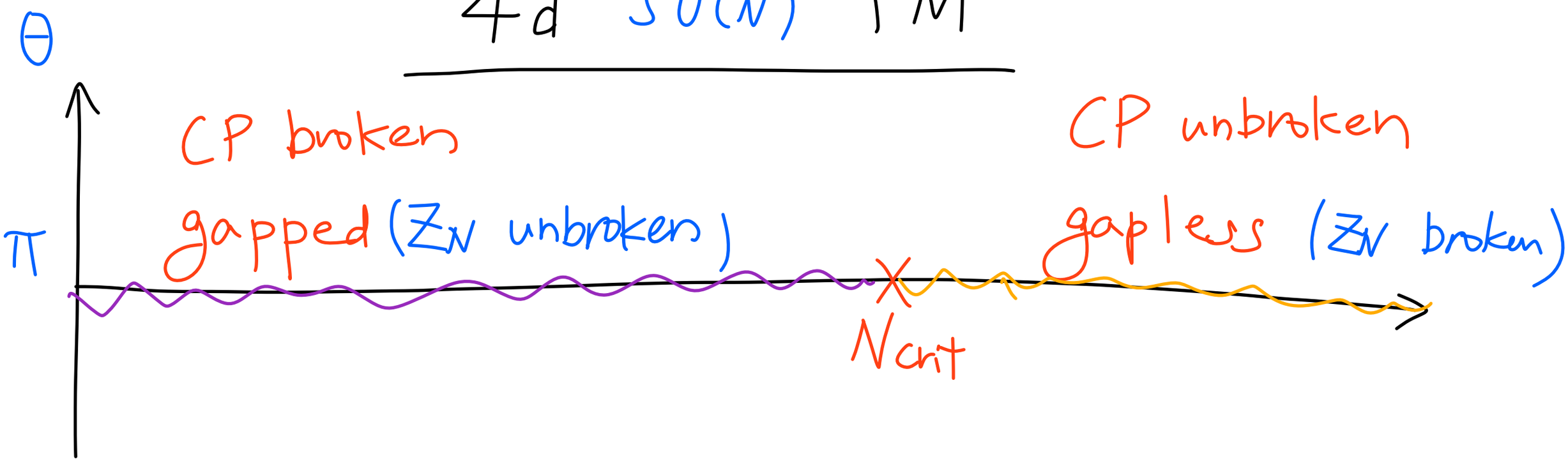

Θ
0
$\frac{1}{N}$
Large N
"classical"
Small N
"quantum"


θ
$E = \frac{1}{2}\chi\theta^2\left(1 + \frac{b_2^{(1)}}{N^2}\theta^2 + \frac{b_4^{(1)}}{N^4}\theta^4 + \cdots\right)$
?
$E = M^4\Lambda^4(1-\cos\theta)$
0
$\frac{1}{N}$
Large N
"classical"
N_*
$= 12/11$
Small N
"quantum"

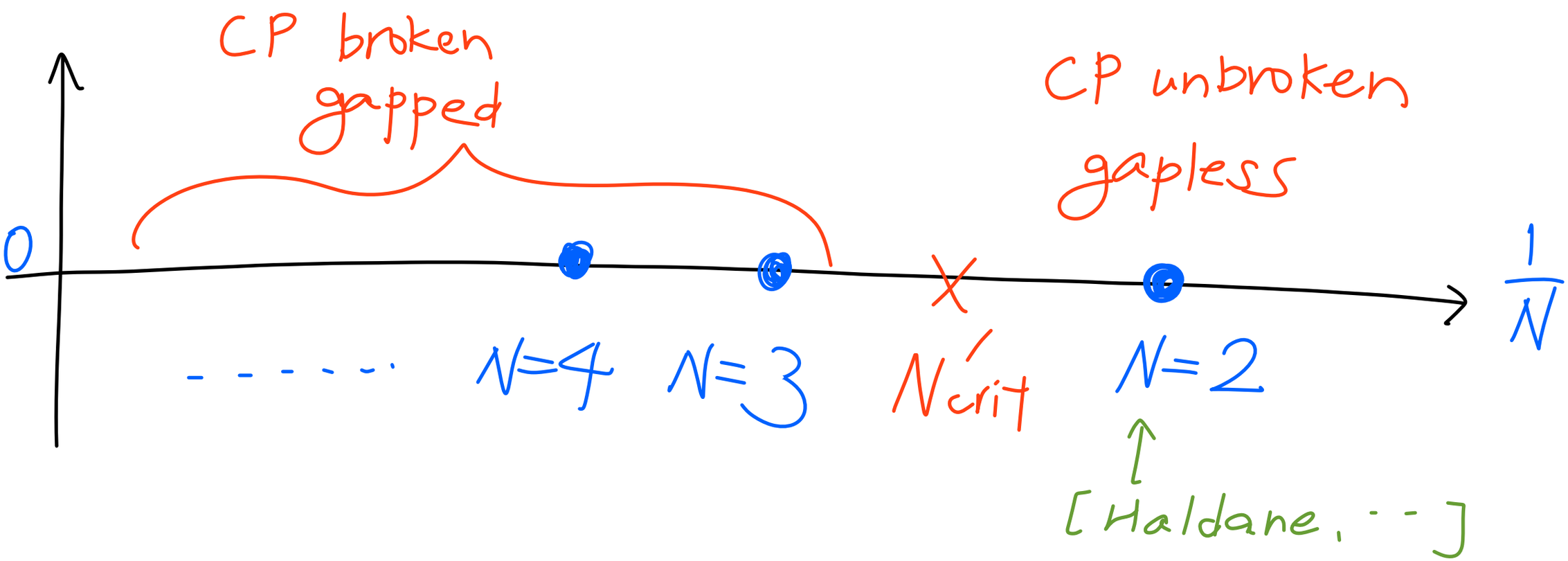
[cf. Z_N–CP mixed anomaly GKKS ('17)]



4d $SU(N)$ YM



2d CP^{N-1} model



4d $SU(2)$ YM + θ-angle

$$E(\theta) - E(0) = \frac{1}{2}\chi\,\theta^2\left(1 + b_2\theta^2 + b_4\theta^4 + \cdots\cdots\right)$$

Compute

Is $N=2$ large (large N) or small (inst.)?

Is CP preserved/broken @ $\theta=\pi$?

gapped/gapless

Lattice

[many relevant references
see the forthcoming paper for refs]

* conceptually "simple"

generate gauge conf. at $\theta = 0$ ← no sign problem

↓

measure top. charge Q

↓

$$\chi = \frac{\langle Q^2 \rangle_{\theta=0}}{V} ,$$

$$b_2 = -\frac{\langle Q^4 \rangle_{\theta=0} - 3 \langle Q^2 \rangle_{\theta=0}^2}{12 \langle Q^2 \rangle_{\theta=0}} ,$$

$$b_4 = \frac{\langle Q^6 \rangle_{\theta=0} - 15 \langle Q^2 \rangle_{\theta=0} \langle Q^4 \rangle_{\theta=0} + 30 \langle Q^2 \rangle_{\theta=0}^3}{360 \langle Q^2 \rangle_{\theta=0}} ,$$

* in practice several subtleties / difficulties

Need statistics ← deviation from Gaussian

$$\left(e^{-\frac{1}{2}\chi\theta^2} \sim Z(\theta) = \sum_Q Z_Q\, e^{iQ\theta} \quad \downarrow \quad Z_Q \sim e^{-Q^2/2\chi} \right)$$

1,75
1,85 $16^3 \times 32$
1,975 $24^3 \times 48$ $(aT_c)^{-2}$ many

β	N_S	N_{T_c}	$(a\,T_c)^2$	$L\,\sigma_{\rm str}^{1/2}$	statistics
1.750	16	4.65	0.0462	4.9	80,100
1.850	16	6.50	0.0237	3.5	71,040
1.975	16	9.50	0.0111	2.4	30,490
1.975	24	9.50	0.0111	3.6	131,830

(Symanzik action, HMC, Bridge++)

Short-Distance Fluctuations

We have fluctuations of size $\sim O(a)$

$\rightarrow$ removed by "smearing"

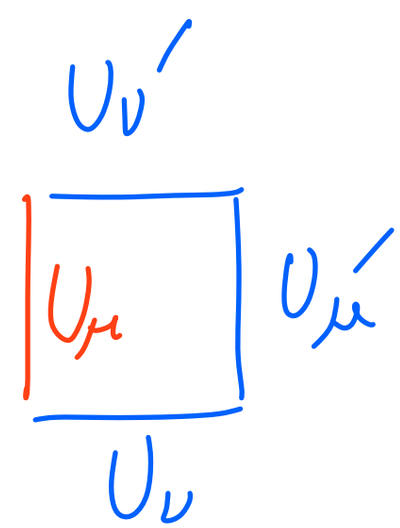
[many different methods, see Alexandrou + ('17) for comparison]

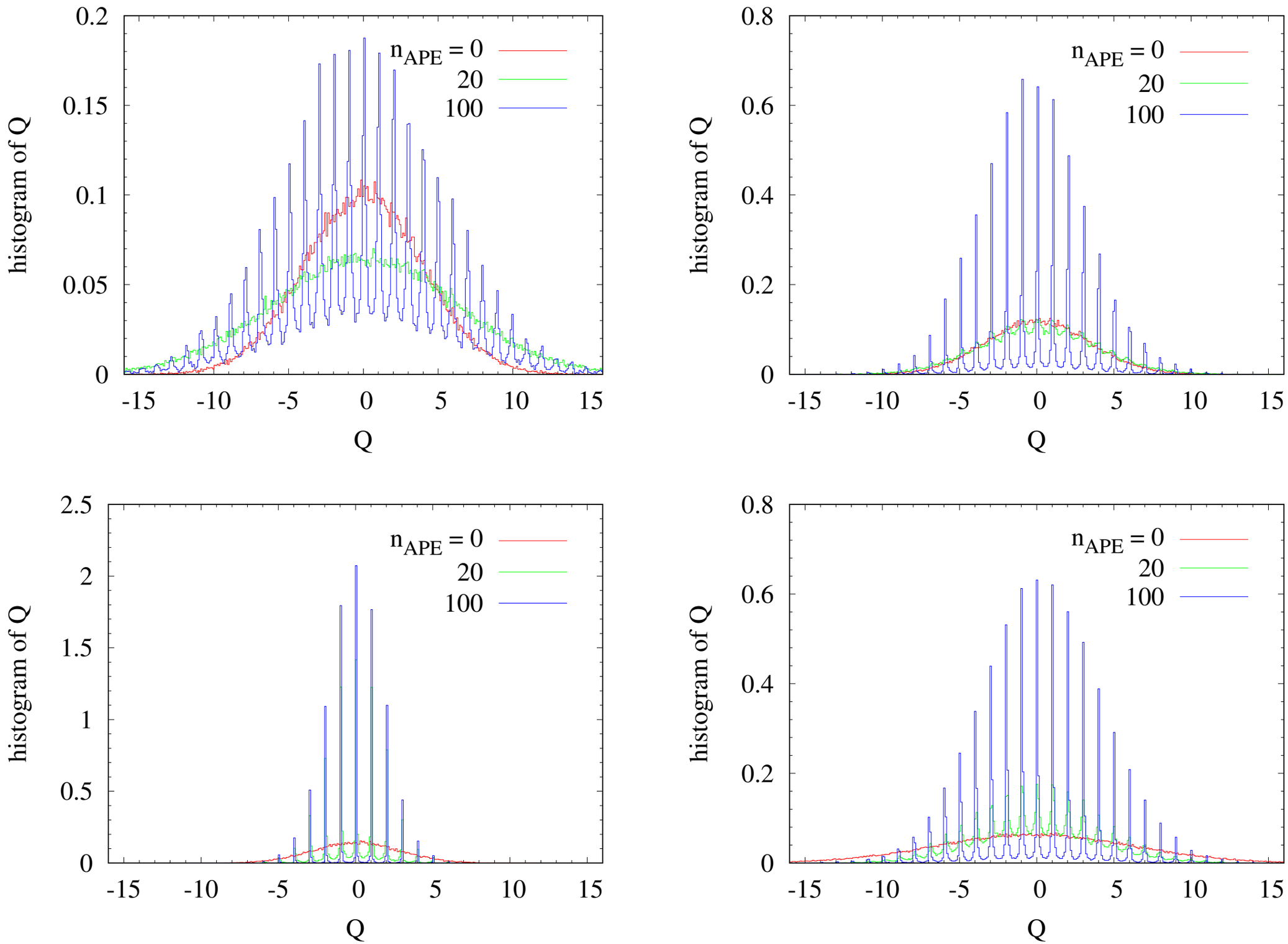
We use APE smearing (& gradient flow)

[Albanese + ('87)]

$$U_\mu^{(\mathrm{new})} = \mathrm{Proj}\left[(1-\rho)\mathrm{U}_\mu^{(\mathrm{old})}(\mathrm{x}) + \rho\,\mathrm{X}_\mu(\mathrm{x})\right] \ ,$$

$$X_\mu(x) = \sum_{\nu\neq\mu}\Big[U_\nu^{(\mathrm{old})}(x)U_\mu^{(\mathrm{old})}(x+\hat{\nu})U_\nu^{(\mathrm{old})\dagger}(x+\hat{\mu}) + U_\nu^{(\mathrm{old})\dagger}(x-\hat{\nu})U_\mu^{(\mathrm{old})}(x-\hat{\nu})U_\nu^{(\mathrm{old})}(x-\hat{\nu}+\hat{\mu})\Big] \ ,$$




histogram of Q
Q
n_{APE} = 0
20
100
0.2
0.15
0.1
0.05
0
0.8
0.6
0.4
2.5
2
1.5
1
0.5
-15
-10
-5
5
10
15

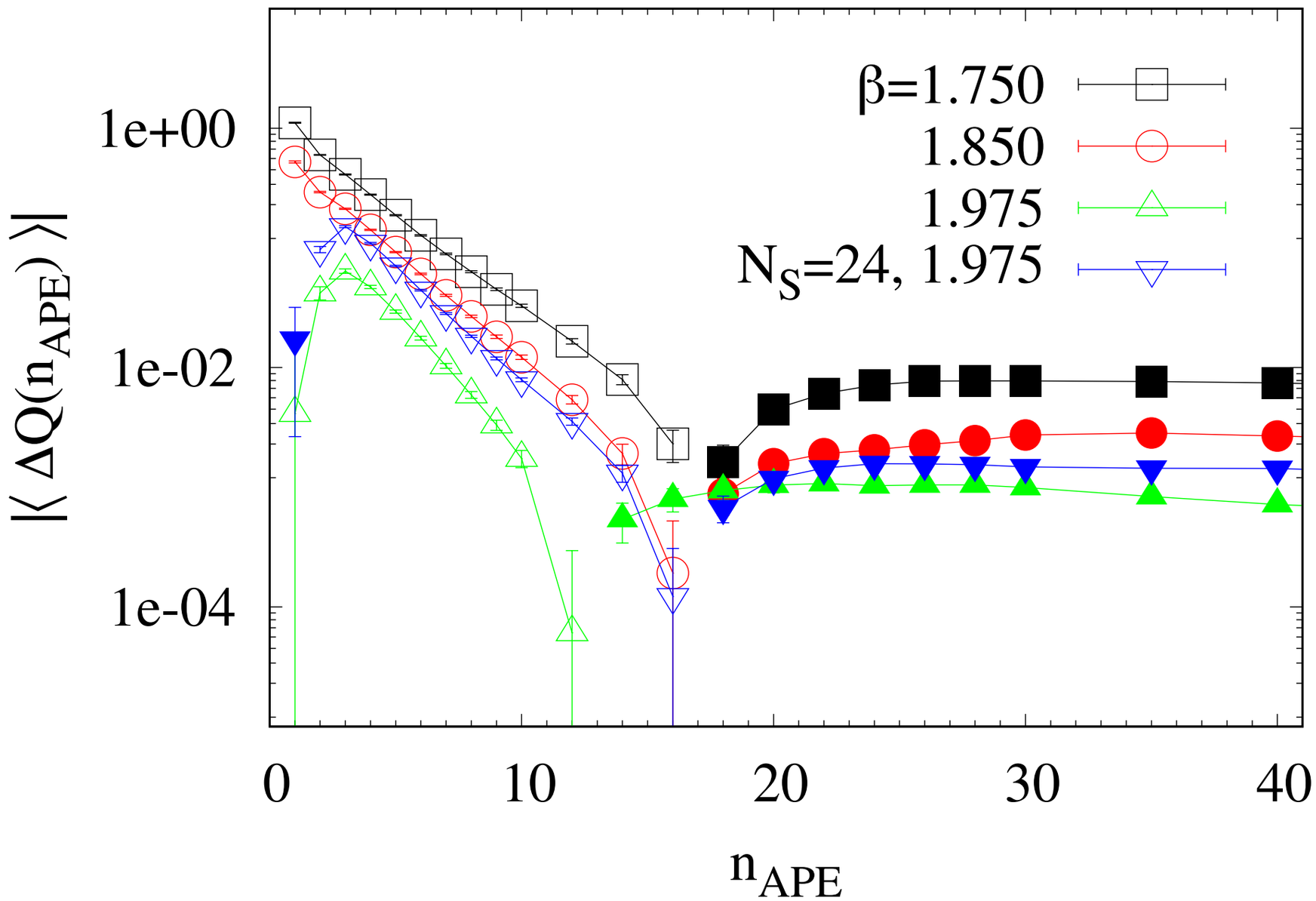
Smearing

successfully removes short-distance fluctuations ← first "phase" n_{APE} small

but also changes physical instanton configurations ← second "phase" n_{APE} large

"falling through lattice"

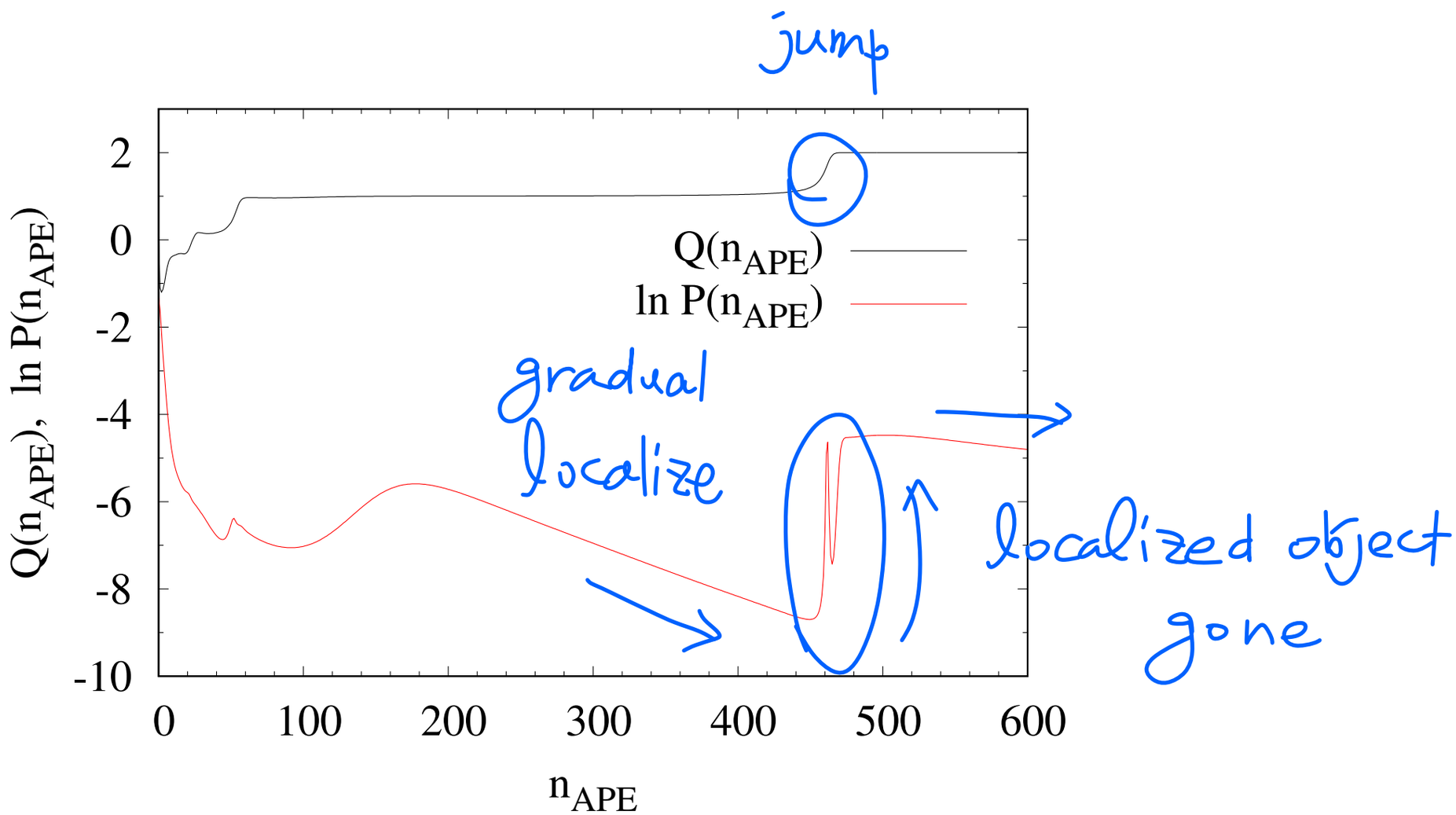
[Bilson-Thompson, Leinweber, Williams, Dunne ('03)]

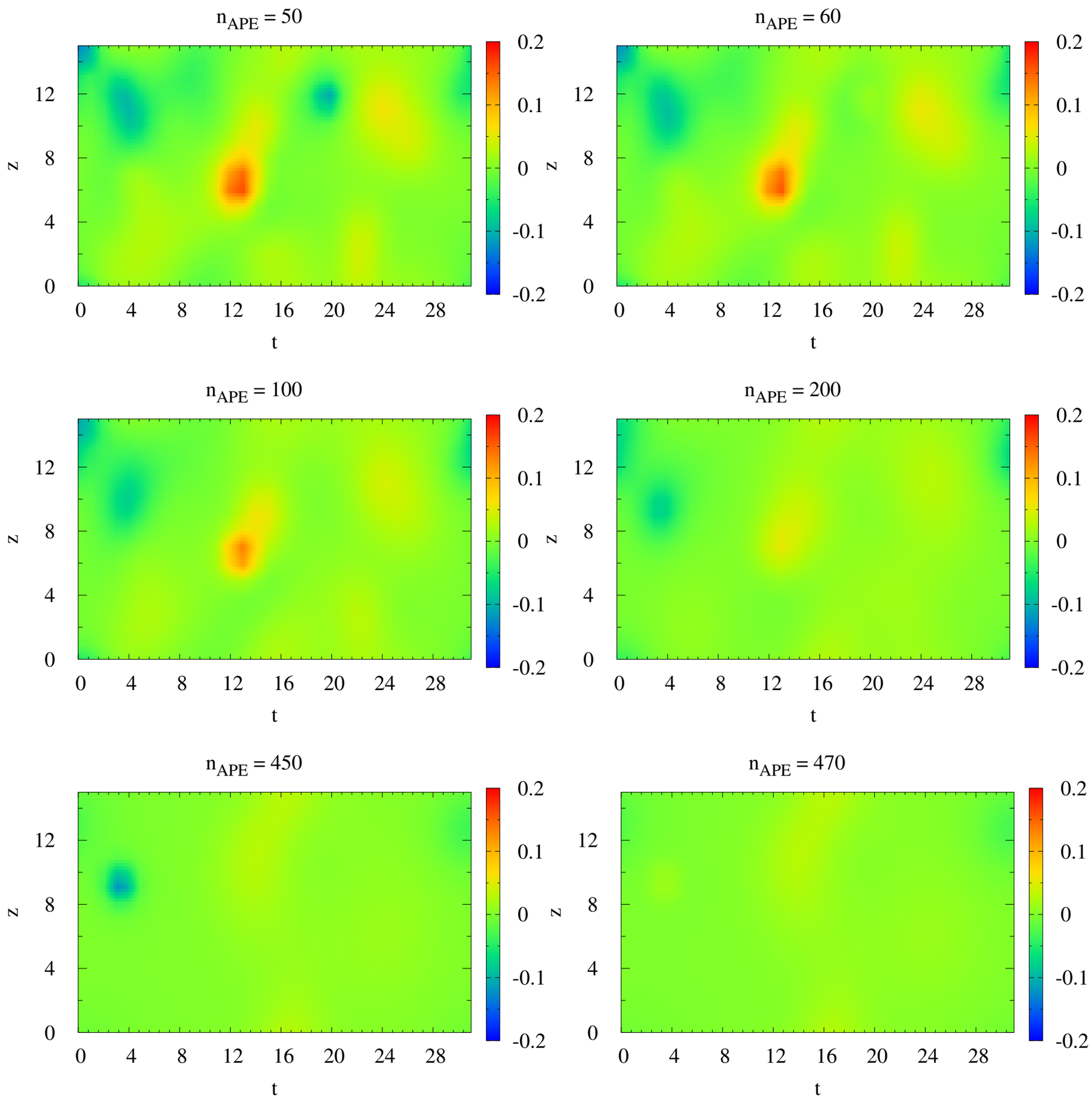

β=1.750
1.850
1.975
N_S=24, 1.975
1e+00
1e-02
1e-04
|⟨ΔQ(n_APE)⟩|
0
10
20
30
40
n_APE

$$P(n_{\mathrm{APE}}) = \frac{1}{N_{\mathrm{site}}} \frac{\left(\sum_x q(x, n_{\mathrm{APE}})^2\right)^2}{\sum_x q(x, n_{\mathrm{APE}})^4},$$

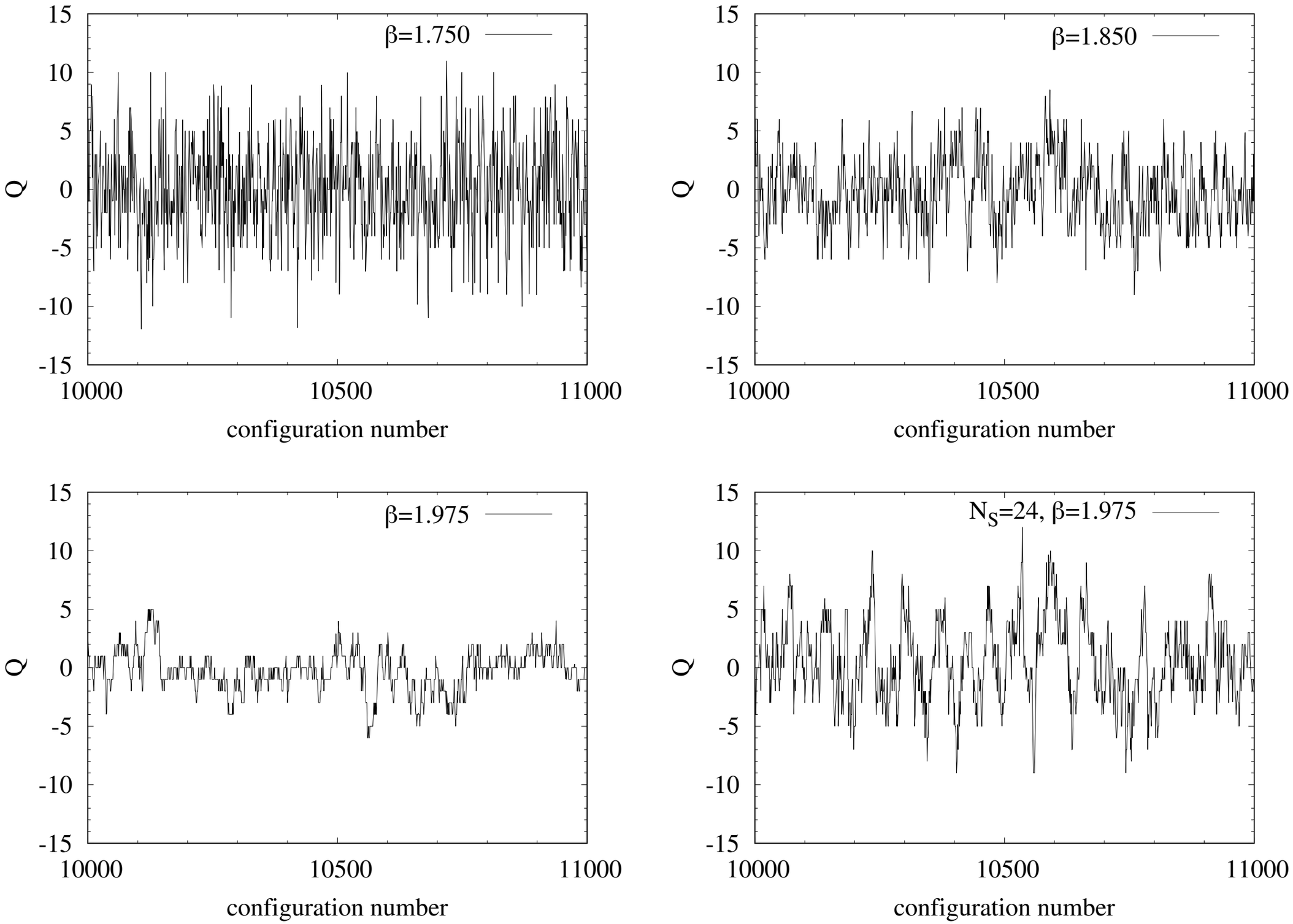
$\nearrow$ max $P = 1$ (uniform)

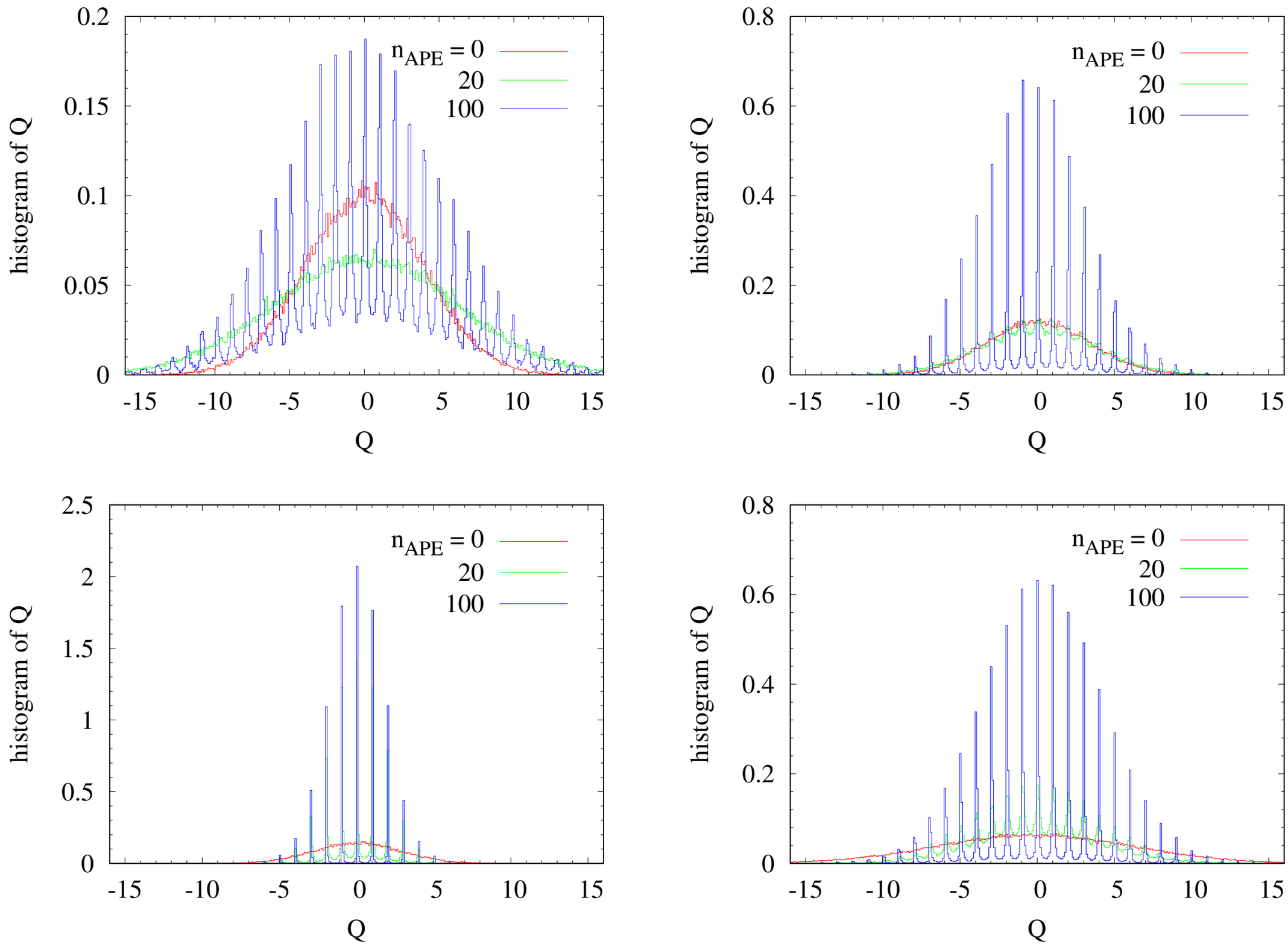
$\searrow$ min $P = \frac{1}{N_{site}}$ (δ function)

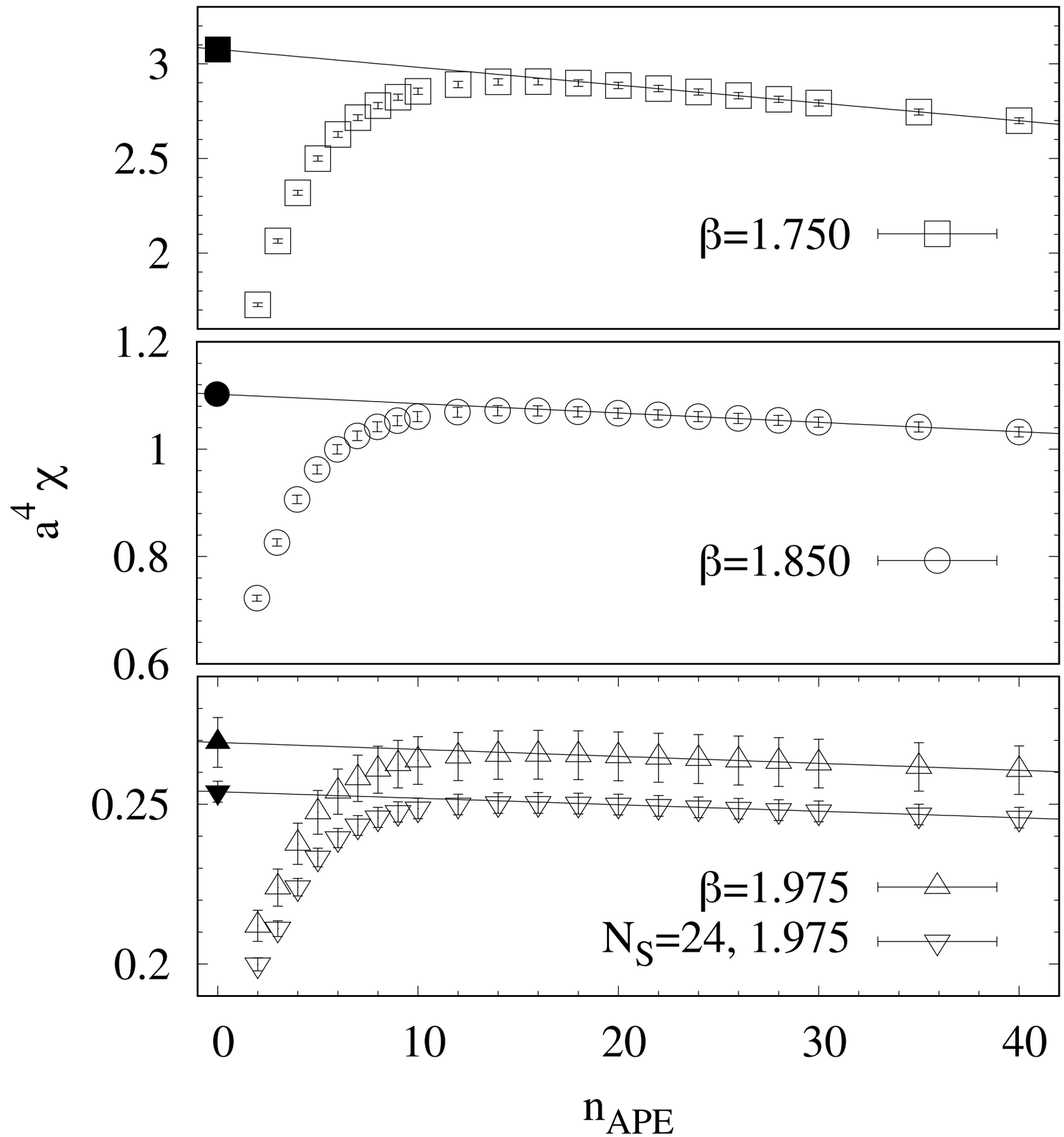


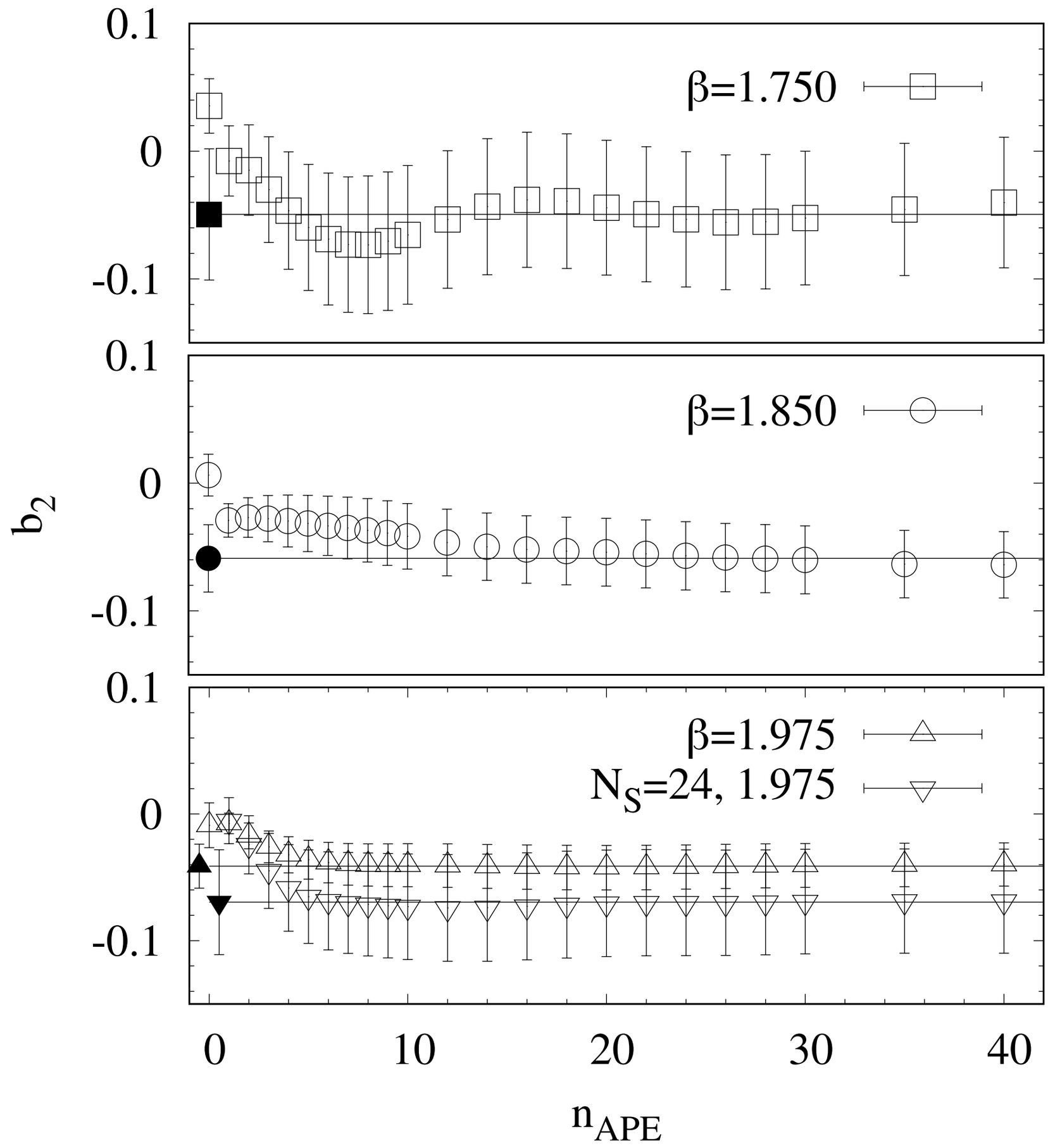

n_{APE} = 50
n_{APE} = 60
n_{APE} = 100
n_{APE} = 200
n_{APE} = 450
n_{APE} = 470
z
t
0
4
8
12
16
20
24
28
0.2
0.1
0
-0.1
-0.2

Result


β=1.750
β=1.850
β=1.975
N_S=24, β=1.975
Q
configuration number
15
10
5
0
-5
-10
-15
10000
10500
11000


histogram of Q
Q
n_{APE} = 0
20
100
0.2
0.15
0.1
0.05
0
0.8
0.6
0.4
2.5
2
1.5
1
0.5
-15
-10
-5
5
10
15

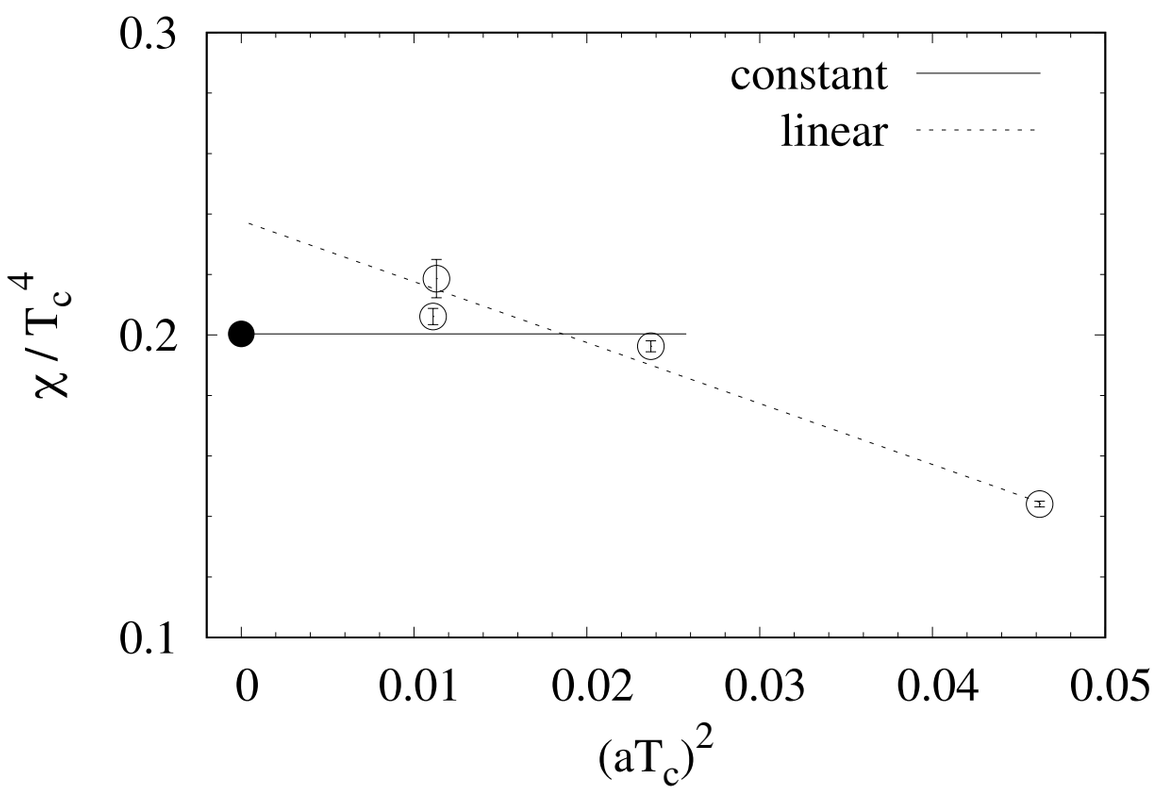

3
2.5
2
1.2
1
0.8
0.6
0.25
0.2
$a^4\chi$
β=1.750
β=1.850
β=1.975
N_S=24, 1.975
0
10
20
30
40
n_{APE}

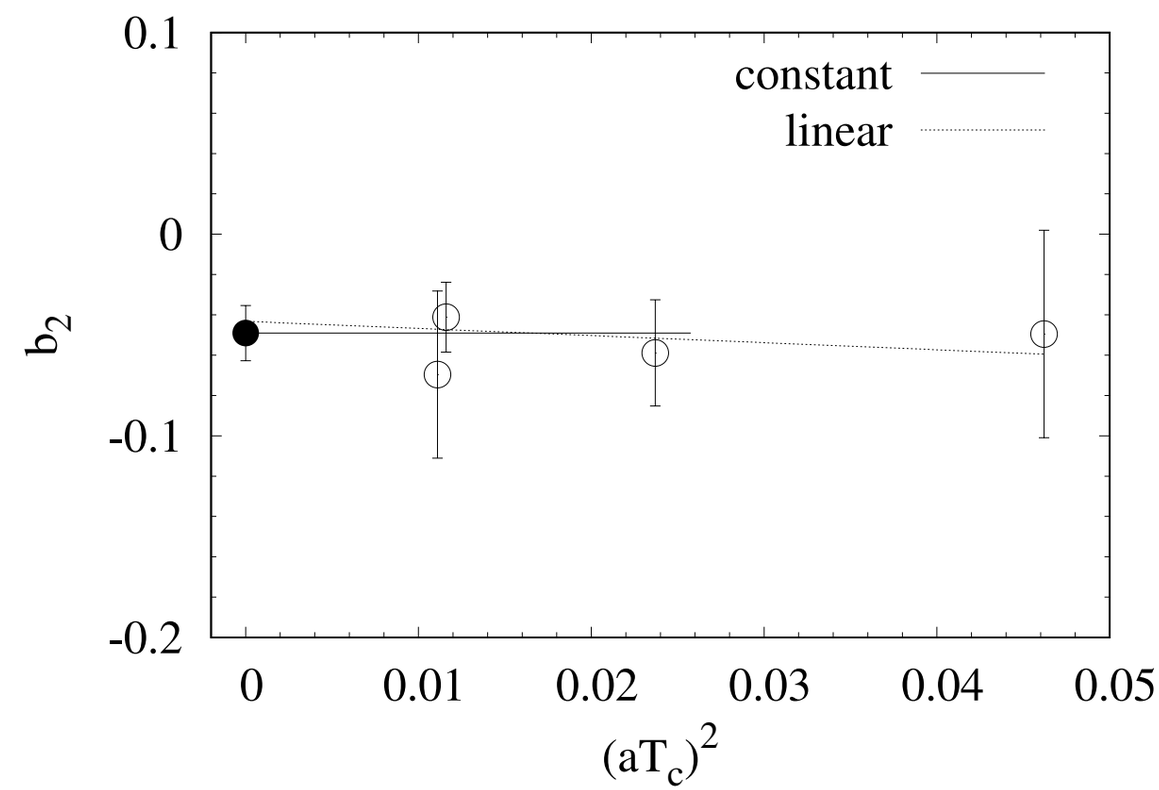

0.1
0
-0.1
β=1.750
0.1
0
-0.1
β=1.850
0.1
0
-0.1
β=1.975
N_S=24, 1.975
b_2
0
10
20
30
40
n_APE

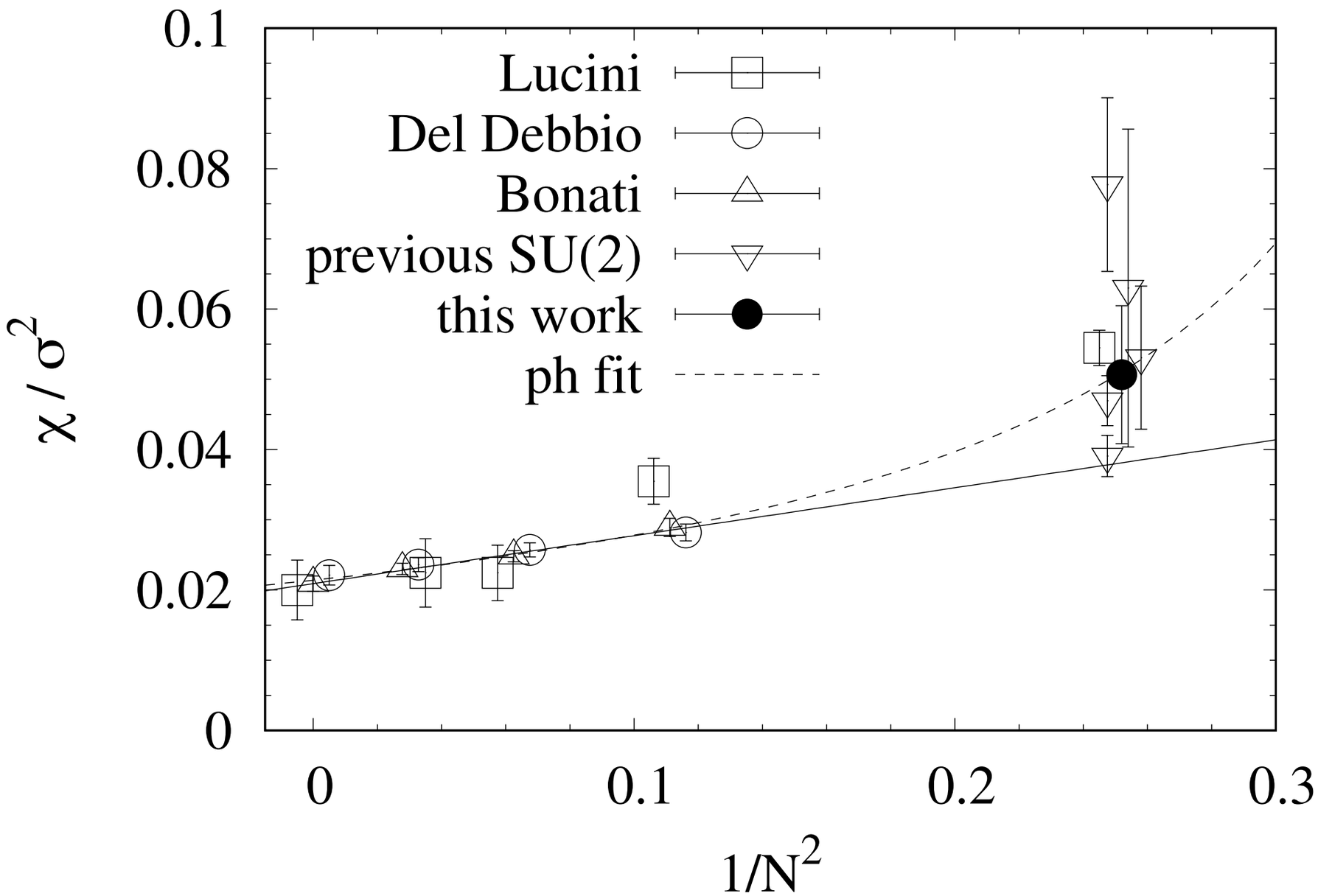
$$\frac{\chi}{T_c^4} = 0.200(39) \ , \qquad \frac{\chi^{1/4}}{T_c} = 0.674(31) \ , \qquad b_2 = -0.049(20) \ ,$$

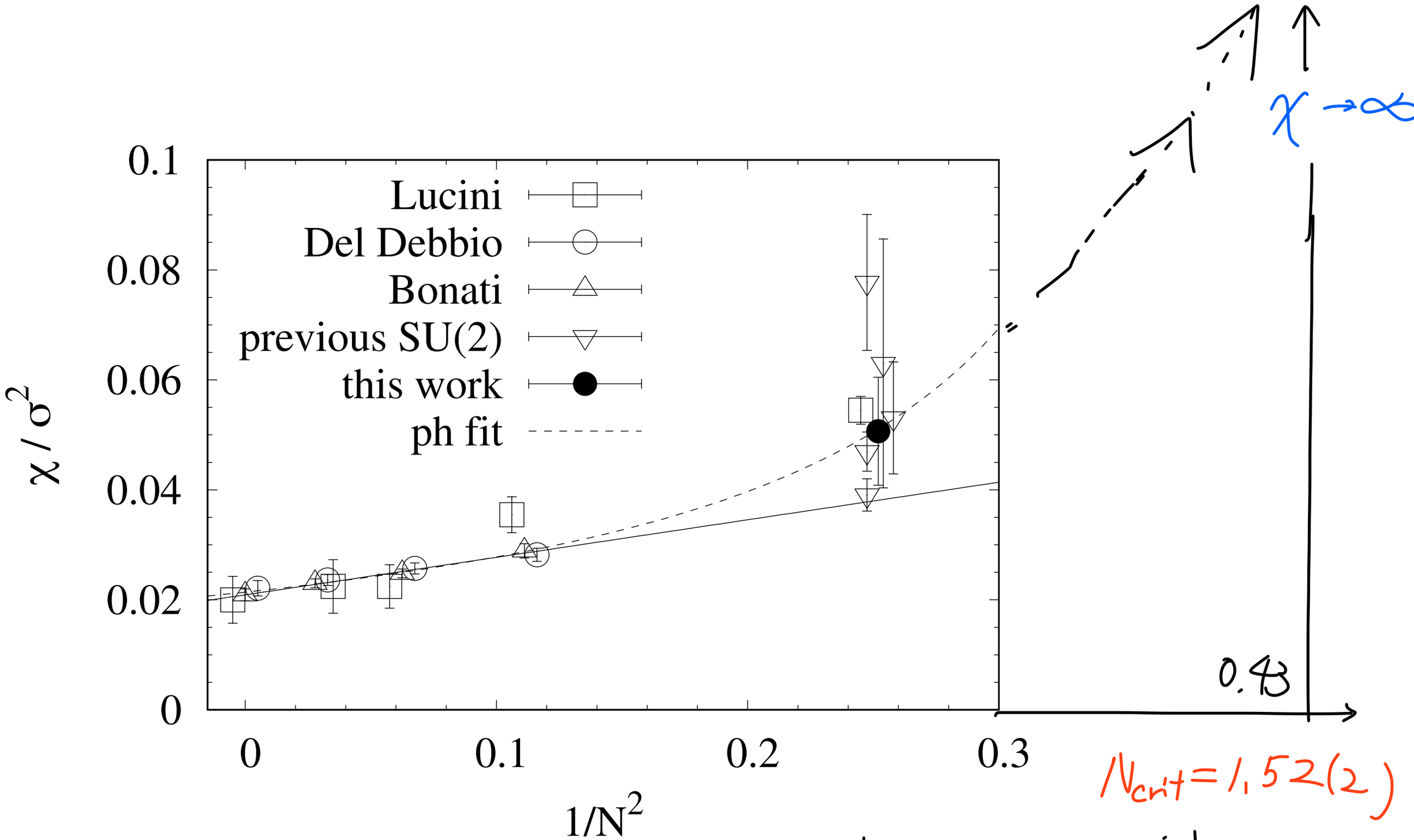
seems to be the first determination of b_2

[cf. Bonanno, Bonati, D'Elia (18) $b_4 = 6(2) \cdot 10^{-4}$]






Lucini
Del Debbio
Bonati
previous SU(2)
this work
ph fit
0.1
0.08
0.06
0.04
0.02
0
χ / σ^2
0
0.1
0.2
0.3
$1/N^2$

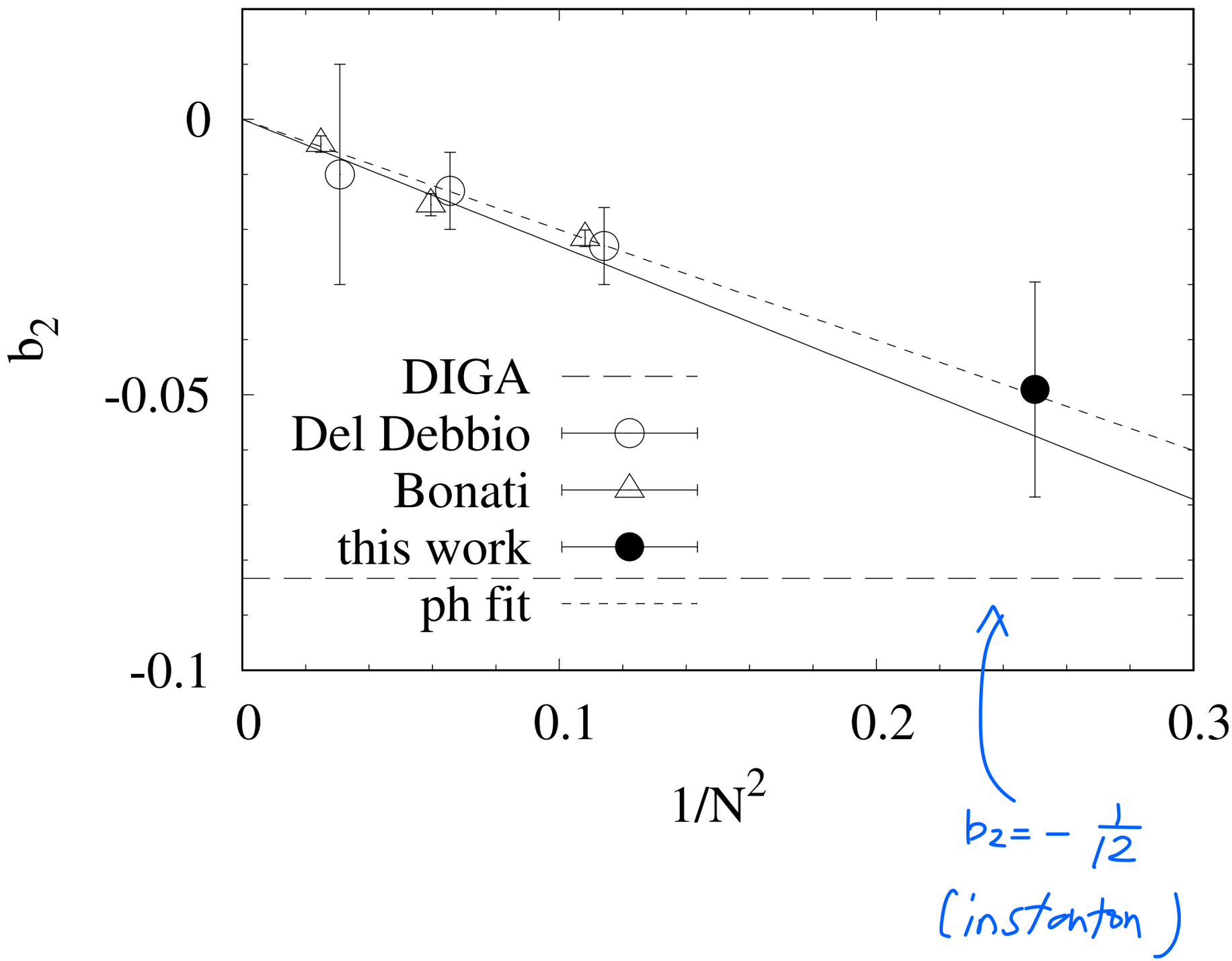


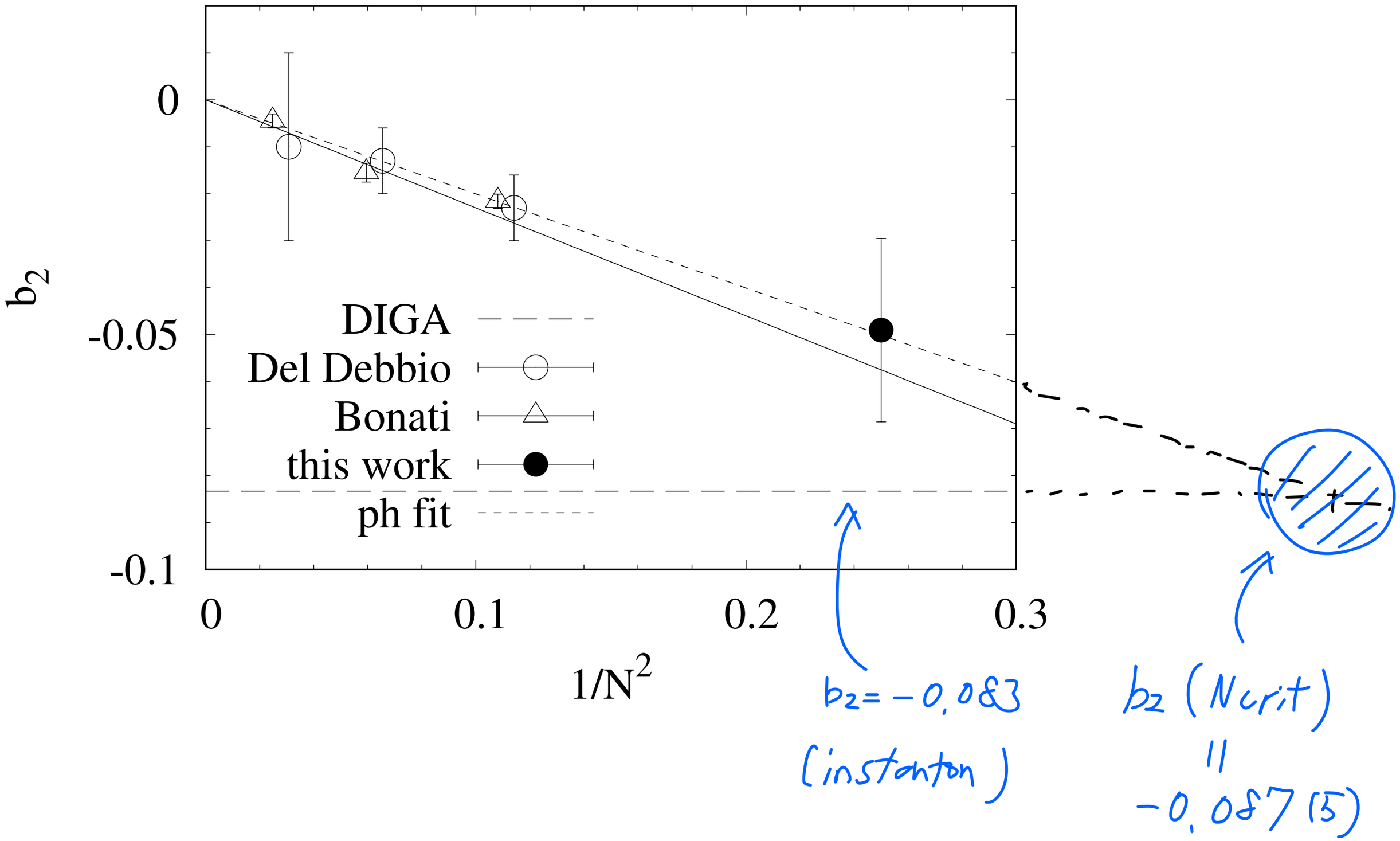
$1/N_{crit} = 1.52(2)$

when fitted with

[cf. Lüscher (182) for 2d CP^{N-1} model]

$\left(\begin{array}{l}\chi \to \infty \\ \text{at} \\ N = N_{crit}\end{array}\right) \rightsquigarrow \dfrac{\chi}{\sigma} = \left(\dfrac{\chi}{\sigma}\right)_{N=\infty} \dfrac{N^2}{N^2 - N_{crit}^2}$


0
-0.05
-0.1
0
0.1
0.2
0.3
b_2
$1/N^2$
DIGA
Del Debbio
Bonati
this work
ph fit
$b_2 = -\frac{1}{12}$
(instanton)


0
-0.05
-0.1
0
0.1
0.2
0.3
b_2
$1/N^2$
DIGA
Del Debbio
Bonati
this work
ph fit
$b_2 = -0.083$
(instanton)
$b_2(N_{crit})$
=
$-0.087(5)$

Summary

* 4d SU(2) YM : still "large N"

spontaneous CP breaking, mass gap

@ $\theta = \pi$

$$\frac{\chi^{1/4}}{T_c} = 0.674(31), \quad b_2 = -0.049(20)$$

[Quantitatively different from 2d CP^{N-1} model]

* Transition to "small N" happens at

$$N_{crit} \cong 1.52$$